

Economics of Crime Networks

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Abstract

We study a network-based model of criminal activity. Agents' payoffs depend on the number and structure of links among them and are determined in a Nash equilibrium of a crime effort supply game. Unlike much of the existing literature that takes network structure as given, we analyze optimal network structures, defined as maximizing aggregate payoff. Using potential functions, we give necessary and sufficient conditions that guarantee the existence and uniqueness of equilibria with non-negativity constraints on effort. These results can be used to identify optimal networks for given cost and benefit parameter configurations drawing on graph theory and using a computational algorithm that searches over all possible non-isomorphic networks of a given size. Our results can be also used to study, via numerical simulations, the effects of alternative crime reducing policies on the network structure and crime level - removing agents, removing links or varying the probability of apprehension.

PRELIMINARY AND INCOMPLETE

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JEL Classifications:

1 Introduction

Recently there has been considerable theoretical work and some emerging empirical studies integrating social networks, defined as graphs with nodes being economic agents and edges being various links connecting them, into economic modeling (Jackson, 2003, 2004, 2006, 2007, 2008). Various economic applications have benefited from this approach – these include analyses of delinquency, crime, prisons, job search, social norms, human capital investment and social mobility, among many others. However, much of the discussion of networks in economic contexts continues the traditions from sociology or psychology and takes the network as exogenous.¹ Clearly some networks are and should be treated as exogenous to an actor’s decision process. For example, one does not choose one’s relatives, and kin relations and other fixed social relationships have constituted an important application of network theory. However, in principle, there is no reason why the network structure itself should not be part of the decision made by economic agents. Our approach here is to model the network structure as endogenously emerging from solving an economic optimization problem.

We develop a formal network-based model of criminal activity. Social network analysis is likely to be more relevant in situations where markets fail as a consequence of high transaction costs caused by asymmetric information, limited enforcement, externalities and the like. Crime is a prime example of economic activity conducted in such environment – illegal organizations cannot rely on official means of enforcing contracts or sharing information, thus costly alternative mechanisms for performing their operations must be used. Institutions based on non-anonymous interaction such as networks naturally provide such a mechanism.

In our model, agents’ payoffs (net incomes) depend on the number and structure of links among them. The links can be viewed as information channels, indicators of ability to meet, etc. The payoffs also depend on individually chosen actions – “crime effort” by each network member. Efforts are determined in a Nash equilibrium of a simultaneous move game. We analyze the optimal network structures (that is, the pattern of links among agents) that maximize the aggregate payoff of the networked group and the associated aggregate crime level.

Our goal is to shed light on the following questions. What parametric assumptions ensure the existence of Nash equilibrium with non-negative effort choices? Is the equilibrium unique? Are equilibria interior or feature ‘corners’, i.e., agents supplying zero effort? For given cost and benefit parameter values in the model for which a Nash equilibrium in efforts exists, what types of network structures maximize total payoff and what is the associated criminal activity level?

We derive theoretical results that guarantee the existence and uniqueness of Nash equilibria for both cases of interior and corner solutions in effort choice. We obtain these results using the potential function associated with the individual payoff function. We show that as long as the parameter that controls the strength of the congestion effect, is greater or equal to the parameter that determines the positive effect of being connected to other agents, there exists an equilibrium. We also show that the necessary and sufficient conditions for the uniqueness of this solution depends on the relative strength of the costs and benefits of being in the network and the minimum eigenvalue of a matrix related to the adjacency matrix of the network. We also

¹There is a small literature that considers network choice, e.g., Hojman and Szeidl (2006) look at dyadic decisions that lead to an equilibrium network.

derive some specific results regarding the total crime effort and total payoff. We compare the individual optimization problem with a planner’s problem and show that the full (completely connected) network maximizes the total criminal effort level as well as the overall profit if a social planner maximizes overall profits. This is not the case in a Nash equilibrium where total payoff may be maximized for a different network structure. We also show that the individual Nash equilibrium payoff is quadratic in the equilibrium effort level and that agents who exert zero effort in equilibrium are linked to agents who supply less effort in comparison to the neighbors of an agent with positive effort in equilibrium.

Analytically, we model an N -member network. Each of the N agents has a payoff function, U_i that represents the net value or benefit (“income” net of costs) that the agent obtains from interacting with others. The network structure, G is a crucial determinant of the agent’s benefits and costs. In our application to criminal networks we assume that an agent’s payoff depends on the level of criminal activity (“effort”) the agent performs, e_i and the number of connections he has with other agents. The number and pattern of links affects on the one hand the income (benefit) of an agent (e.g., through cooperation) and, on the other hand, his costs (e.g., by raising the probability of apprehension). The total amount of crime activity is determined in Nash equilibrium whereby each agent $i = 1, \dots, N$ simultaneously chooses his effort level, e_i , taking the efforts of all other network members, e_j , $j \neq i$ as given. To derive the necessary and sufficient conditions for existence and uniqueness of the Nash equilibrium, we adopt to our setting and apply techniques introduced by Bramoullé, Kranton and D’Amours (2011). They derive conditions for existence and uniqueness of equilibria using potential functions and the minimum eigenvalue of the network adjacency matrix. Specifically, we embed the adjacency matrix, G into another matrix that contains the parameters of the model and derive conditions for uniqueness and existence using this modified matrix.

Unlike much of the existing literature which takes the network structure as unmodelled or exogenously given, we employ a computational algorithm developed in applied mathematics (see Appendix 1) that searches over all possible *non-isomorphic* networks, that is, all networks that cannot be obtained from each other by re-labeling nodes. It becomes computationally infeasible very quickly to search over all possible networks without focusing exclusively on non-isomorphic networks. Finding all non-isomorphic networks (or, ‘simple graphs’ as they are also known) is a complex combinatorial problem and currently no computational or theoretical algorithm to solve it for any N exists.² Still, the problem has been solved for small network sizes ($N = 2$ to $N = 11$ players).³ We use the computed data – that is, the list of all networks of a given size that are non-isomorphic – as an input to our search algorithm. For each non-isomorphic network and set of model parameters for which an equilibrium in non-negative efforts exists, we can compute the individual and aggregate crime effort and payoff levels. We are then able to use this information to find the optimal network, defined as the aggregate payoff maximizing network among all non-isomorphic networks of size N) for a large representative set of cost and benefit parameters. We analyze the structure of the optimal network and associated crime

²In fact, this problem belongs to its own class in complexity theory called ‘graph isomorphism complete’ and is thought to be non-verifiable in non-deterministic polynomial time (NP-complete) – see Skiena (1981).

³See McKay (1981) for an early algorithm description as well as B. McKay’s webpage: <http://cs.anu.edu.au/people/bdm/>

levels depending on the parameters as well as the frequency and patterns with which various structures emerge as optimal.

We also intend to use numerical simulations of our model for a representative large set of parameters to assess the relative effectiveness of various possible crime-reducing policies: removing players, removing links or by varying the probability of apprehension. We look at the effect of these policies on the optimal network answering the following questions. For a particular network structure, what constitutes the optimal crime deterrence policy? How does the optimal network structure respond to policy (crime prevention techniques used)? Conversely, given a particular deterrence policy, what is the optimal network structure that arises to minimize the costs inflicted by that policy? What structures and crime levels emerge as the joint outcome? What is necessary to explain the observed change (e.g., Raab and Milward, 2003) in the structure of drug-trafficking networks from relatively hierarchical organizations to more decentralized ones? Preliminary results indicate that in many cases the optimal structures are special networks identified in the theoretical literature: the “line”, the “wheel”, the “star” and the “complete” network (Bala and Goyal, 2000). We also find some evidence of the optimality of “cell” type structures. The policy analysis is able to take into account the optimal network structure emerging as a result of an announced policy, both in the short run holding the network as fixed, and in the long run when the network can be re-optimized. This part of the paper is still in progress.

In terms of related literature, the role of networks in organized crime has began to be studied by criminologists – e.g., see Sarnecki (2001), Bruinsma and Bernasco (2004). Kenny (2007) reviews the recent literature. Other areas of crime have also benefited from more explicit use of network theory, including human trafficking, crime groups formed by youth, and drug distribution (Hughes, 2000; Coles, 2001; Frank, 2001 and Hoffer, 2002). The events of September 11 and the discovery of the “cell”-type network structure of Al Qaeda have spawned policy work devoted to combating terrorist networks (for example, Carley et al., 2001; Krebs, 2001; Raab and Milward, 2003). Kenny (2007) reviews much of the criminology literature that uses some form of network analysis. Easton and Karaivanov (2009) provide a non-technical version of the model studied here and give some simple examples of policy applications.

In economics, Ballester, Calvo-Armengol and Zenou (2004) (hereafter, BCZ) provide one of the first economic treatments of crime networks concentrating on identifying the ‘key player’ in the network: the player whose removal leads to the greatest decline in criminal activity. While our approach borrows many ideas from BCZ (2004), we adopt a very different assumption about how costs and benefits depend on the network structure and total effort level. Additionally, in contrast to BCZ, we do not take the network structure as exogenous but optimize over all possible networks of a given size, as well as across different sizes. Furthermore, we plan to analyze and compare the effects of various alternative crime prevention policies in addition to the “removal of key player” strategy.

The paper is organized as follows. Section 2 describes the model. In Section 3, we introduce the potential function and present necessary and sufficient conditions for existence and uniqueness of Nash equilibria in crime effort supply. Section 4 discusses various properties of the equilibrium and associated crime networks, including results that we derive by comparing the individuals’ effort optimization problem to a planner’s problem. In Section 5 we motivate

and discuss how our theoretical setting can be used to study optimal networks and the crime level in alternative crime-detering policy environments. Section 6 concludes by listing some of the possible challenges and remaining questions for future research.

2 The Model

There are N agents whose interaction we model as a social network (graph): a list of nodes representing individual agents (players) and the links between them. We assume only bidirectional links so that if player i is connected to player j , then the reverse is also true. The network structure can be fully summarized by the ‘adjacency matrix’, G – an N by N matrix with zeros on the main diagonal (by convention) and elements, g_{ij} equal to 1 if players i and j are connected and 0 otherwise. The assumed bidirectional nature of links means that G is symmetric.

For a given network structure, each agent decides on a level of criminal activity, his “effort”, $e_i \geq 0$ to maximize his net income (benefits minus costs). Efforts are picked by the agents simultaneously, in a Nash equilibrium manner. We follow BCZ (2004) and use a quadratic form for the benefit and cost functions. This results in an easy-to-solve linear system for the equilibrium crime levels. An important difference with BCZ is that we assume that benefits increase in the number of links a player has to other players’ efforts while the total amount of crime, that is the sum of others’ efforts, creates a congestion effect increasing one’s costs, e.g., arising from a greater likelihood of being detected.⁴ In addition, we allow for a per-unit effort cost of maintaining a link between players and a ‘standalone’ cost of effort independent of other’s actions.

Specifically, let player i ’s net income, U_i (which in our model coincides with utility) be given by:

$$U_i(G, \mathbf{e}) = y_i(G, \mathbf{e}) - c_i(G, \mathbf{e})$$

where:

$$y_i(G, \mathbf{e}) = e_i \left(1 + \gamma \sum_{j=1}^N g_{ij} e_j \right)$$

and

$$c_i(G, \mathbf{e}) = e_i \left(\pi + \lambda \sum_{j=1}^N e_j + \delta \sum_{j=1}^N g_{ij} \right)$$

and where \mathbf{e} denotes the vector of all agents’ efforts, $\mathbf{e} \equiv (e_1, \dots, e_N)$.

The parameter $\gamma \geq 0$ determines the strength of the benefit from having links with other (active) agents, while the parameter $\lambda \geq 0$ determines the strength of the ‘congestion’ effect. We allow for the possibility of costly link maintenance through the parameter $\delta \geq 0$. The parameter $\pi \in [0, 1)$ determines the standalone unit cost of effort.

The optimal effort choices e_i^* for all $i = 1, \dots, N$ are determined in a Nash equilibrium with

⁴In BCZ (2004) benefits increase in the total amount of crime while costs decrease in the number of connections to other criminals.

each player maximizing their net income U_i taking all other players' effort levels as given and subject to the non-negativity constraints $e_i \geq 0$. In matrix notation, an agent's net income can be written as:

$$U_i(G, \mathbf{e}) = e_i(1 - \pi - \delta G \mathbf{1}) + \gamma e_i \mathbf{G} \mathbf{e} - \lambda e_i \mathbf{e} \mathbf{1} \quad (1)$$

where \mathbf{e} is the N -by-1 vector of efforts and $\mathbf{1}$ is a N -by-1 vector of ones. The resulting first order conditions, in matrix form, are:⁵

$$[\beta_1 \mathbf{I} - \phi_2 G] \mathbf{1} - [(\mathbf{J} + \mathbf{I})\phi_1 - G] \mathbf{e} \leq \mathbf{0} \text{ with equality if } e_i > 0 \quad (2)$$

where $\phi_1 \equiv \frac{\lambda}{\gamma}$, $\phi_2 \equiv \frac{\delta}{\gamma}$, $\beta_1 \equiv \frac{1-\pi}{\gamma}$ and where \mathbf{I} is the N -by- N identity matrix and \mathbf{J} is an N -by- N matrix of ones. At equality, (??) form a linear system the solution of which (when it exists) is the equilibrium vector of crime levels, \mathbf{e} for the interior solution case.⁶

Clearly, as long as $\det((\mathbf{J} + \mathbf{I})\phi_1 - G) \neq 0$ which happens on a set of Lebesgue measure zero \mathcal{Z} , the linear system

$$[\beta_1 \mathbf{I} - \phi_2 G] \mathbf{1} - [(\mathbf{J} + \mathbf{I})\phi_1 - G] \mathbf{e} = \mathbf{0} \quad (3)$$

has a unique solution. (Note that $\beta_1 \mathbf{I} - \phi_2 G \neq \mathbf{0}$ since $\beta_1 > 0$ and the diagonal elements of G are zeros.) We assume $\phi_1 \notin \mathcal{Z}$ which is the generic case. However, it is possible that (some of) the non-negativity constraints $e_i \geq 0$ are violated at such solution. Using the arguments in BCZ (2004) Proposition 1, if $\phi_1 \notin \mathcal{Z}$ is large enough and ϕ_2 is small enough, the system (??) has a solution consisting of strictly positive crime levels, e_i for all i . BCZ look only at parametrizations which satisfy these conditions and thus solve for the Nash equilibrium in their setting by treating all FOCs in (??) as equalities.

In general, there is no guarantee that the solutions, e_i^* will be interior. Thus, we analyze a more general class of solutions in which we constrain e_i to be non-negative. Effectively, we are looking for a Nash equilibrium in efforts subject to non-negativity constraints. That is, given the crime efforts of everyone else, agent i should have no incentive to vary his effort, unless he is constrained, in which case he might like to reduce his e_i but this is infeasible.

Because of the non-negativity constraints we cannot find the Nash equilibrium efforts e_i by simply solving the linear system of the FOCs taken as equalities, (??). Instead, we adopt a general approach which allows us to obtain the Nash equilibria with non-negativity constraints for a wider range of parameter values for which a non-negative solution to (??) exists. More specifically, we prove the following:

Proposition 1. *The non-negativity constrained Nash equilibrium in crime effort levels, e_i is given by the solution to the following quadratic programming problem⁷*

$$\begin{aligned} \min_{\mathbf{e}} \mathbf{e}^T \{ [(\mathbf{J} + \mathbf{I})\phi_1 - G] \mathbf{e} - (\beta_1 \mathbf{I} - \phi_2 G) \mathbf{1} \} \\ \text{s.t. } [\phi_1 (\mathbf{I} + \mathbf{J}) - G] \mathbf{e} \geq (\beta_1 \mathbf{I} - \phi_2 G) \mathbf{1} \text{ and } \mathbf{e} \geq \mathbf{0} \end{aligned} \quad (4)$$

Proof:

⁵The inequality sign is interpreted element by element.

⁶Appendix 2 gives an example of the solution for $N = 4$.

⁷The inequality signs apply to each vector element separately.

Minimizing $\mathbf{e}^T[(\mathbf{J}+\mathbf{I})\phi_1-G]\mathbf{e} - \mathbf{e}^T(\beta_1\mathbf{I}-\phi_2G)\mathbf{1}$ is equivalent to minimizing $\sum e_i\{[(\mathbf{J}+\mathbf{I})\phi_1-G]\mathbf{e}-(\beta_1\mathbf{I}-\phi_2G)\mathbf{1}\}_i$. Notice first that if $e_i > 0$ is optimal, $\{[(\mathbf{J}+\mathbf{I})\phi_1-G]\mathbf{e}-(\beta_1\mathbf{I}-\phi_2G)\mathbf{1}\}_i > 0$ will not minimize the objective function, i.e. $\{[(\mathbf{J}+\mathbf{I})\phi_1-G]\mathbf{e}-(\beta_1\mathbf{I}-\phi_2G)\mathbf{1}\}_i$ must be zero at the optimum (the FOC of the agent is satisfied with equality). On the other hand, if $e_i = 0$ is optimal (the agent is constrained), it must be true, by the inequality constraint, that $\{[(\mathbf{J}+\mathbf{I})\phi_1-G]\mathbf{e}-(\beta_1\mathbf{I}-\phi_2G)\mathbf{1}\}_i \geq 0$ i.e. the agent does not want to increase her crime effort from 0. Thus, the quadratic programming problem (??) gives the solution to the Nash equilibrium with non-negativity constraints. ■

In general, an analytical characterization of the Nash equilibrium crime levels is not possible. This calls for a numerical approach. Still, for the special case when there are no link maintenance costs, $\delta = 0$ we are able to establish a number of results regarding existence and uniqueness of Nash equilibria in agents' effort choices. Specifically, we adopt the approach used by Bramoullé, Kranton and D'Amours (2011) (hereafter, BKD) which uses the game theoretic concept of potential functions.

From now on in this section, assume $\delta = 0$ and write the net income function as follows:

$$U_i(G, \mathbf{e}) = \left\{ e_i(1 - \pi) - \lambda \sum_{j=1}^N e_i e_j + \gamma \sum_{j=1}^N g_{ij} e_i e_j \right\}$$

From the first order conditions (??), the optimal effort level for agent i at an interior solution is:

$$e_i^* = \left\{ \beta - \phi \sum_{j=1}^N a_{ij} e_j^* \right\}$$

where $\beta \equiv \frac{1-\pi}{2\lambda}$, $\phi \equiv \frac{\gamma}{2\lambda}$ and where $\{a_{ij}\}_{i,j=1}^N$ are the elements of the N -by- N matrix $\mathbf{A} = \left[\frac{\lambda}{\gamma}(\mathbf{J} - \mathbf{I}) - \mathbf{G} \right]$.⁸ Therefore, taking into account the non-negativity constraints on effort, agent i 's best response function can be written as $\max\{0, \beta - \phi \sum_{j=1}^N a_{ij} e_j\}$. Normalizing efforts \mathbf{e} by $\beta \neq 0$ by calling $\hat{\mathbf{e}} = \frac{\mathbf{e}}{\beta}$ the best response function takes the following form:

$$\hat{e}_i = f_i(\mathbf{e}, \mathbf{A}) \equiv \max\{0, 1 - \phi \sum_{j=1}^N a_{ij} \hat{e}_j\} \quad (5)$$

Therefore, a *Nash equilibrium* with non-negativity constraints is a vector $\hat{\mathbf{e}}$ that simultaneously satisfies $\hat{e}_i = f_i(\hat{\mathbf{e}}, \mathbf{A})$ for all $i = 1, \dots, N$. Another way to interpret the matrix \mathbf{A} is by looking

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That is, we have $a_{ij} = \left[\frac{\lambda}{\gamma}(\mathbf{J} - \mathbf{I}) - \mathbf{G} \right]_{ij}$ and

$$\mathbf{J} - \mathbf{I} = \begin{pmatrix} 0 & 1 & 1 & \dots \\ 1 & 0 & 1 & \dots \\ 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{N \times N}$$

at its components:

$$\begin{cases} a_{ij} = \frac{\lambda}{\gamma} - 1 & \text{if } ij \in G \\ a_{ij} = \frac{\lambda}{\gamma} & \text{if } ij \notin G \\ a_{ij} = 0 & \text{if } i = j \end{cases}$$

In other words, \mathbf{A} is obtained from the network's adjacency matrix G and represents the network with a weighted symmetric matrix in which all agents are connected but the weight of the links between them is different depending on whether or not they are connected in G . Clearly, depending on the values of λ, γ , the elements a_{ij} for $i, j = 1, \dots, N$ could be positive, zero or negative. If $a_{ij} \geq 0$, we have a game of individual efforts being strategic substitutes and when $a_{ij} \in [-1, 0]$ we have a game of strategic complements.

As long as the best response function is a continuous function from $[0, 1]^n$ to itself, a standard fixed point theorem argument guarantees that an equilibrium exist. The following Lemma states the necessary condition for existence of Nash equilibria with non-negative effort levels.

Lemma 1. *The best response function has a fixed point if and only if $\phi < 1/2$.*

Proof: The best response function $f_i(\mathbf{e}, \mathbf{A}) = \max\{0, 1 - \phi \sum_{j=1}^N a_{ij} \hat{e}_j\}$ has a fixed point by Brouwer's Theorem if and only if it is a continuous function from $[0, 1]^n$ to itself. Note that f_i is non-negative by construction, so we only need to insure that $f_i \leq 1$. A necessary condition for this to hold is $a_{ij} \geq 0$, that is $\lambda \geq \gamma$ which implies $\phi \leq 1/2$. To show that this is also a sufficient condition, note that if f_i is best response, then $\hat{e}_i = 1 - \phi \sum_{j=1}^N a_{ij} \hat{e}_j$ if $\hat{e}_i > 0$ and if $1 - \frac{\phi}{\beta} \sum_{j=1}^N a_{ij} \hat{e}_j \leq 0$, then $\hat{e}_i = 0$. We have that $1 - \phi \sum_{j=1}^N a_{ij} \hat{e}_j \leq 1$ implies $\phi \sum_{j=1}^N a_{ij} \hat{e}_j \geq 0$ i.e., $\frac{\gamma}{2\lambda} \sum_{j \neq i} (\frac{\lambda}{\gamma} - g_{ij}) \hat{e}_j \geq 0$, or $\lambda \sum_{j \neq i} \hat{e}_j \geq \gamma \sum_{j \neq i} g_{ij} \hat{e}_i$. Note that $\sum_{j \neq i} \hat{e}_j \geq \sum_{j \neq i} g_{ij} \hat{e}_i$ for all networks and with equality for the full network. Therefore $\lambda \geq \gamma$ is the sufficient condition that guarantees the f_i has a fixed point for any network. ■

From now on, call "active" those agents whose effort levels are strictly positive, i.e., the set of agents $C = \{i = 1, \dots, N | e_i > 0\}$. Similarly, call "inactive" any agents with $e_i = 0$. Let \mathbf{e}_C be the vector of the effort levels of the active agents, \mathbf{A}_C be the subgraph connecting the active agents and $\mathbf{A}_{N-C, C}$ be the subgraph that connects active to inactive agents.

Proposition 2. *An effort profile \vec{e} with active agents C is a Nash equilibrium if and only if:*

$$\begin{aligned} (i) \quad & (\mathbf{I} + \phi \mathbf{A}_C) \mathbf{e}_C = \mathbf{1} \\ (ii) \quad & \phi \mathbf{A}_{N-C, C} \mathbf{e}_C \geq \mathbf{1} \end{aligned}$$

where \mathbf{I} is the $N \times N$ identity matrix and $\mathbf{1}$ is a vector of ones.

Proof: Proposition 2 represents the best response function using matrix notation. Note that, from the first-order conditions, we have:

$$\begin{cases} \hat{e}_i = 1 - \phi \sum a_{ij} \hat{e}_j & \text{if } \hat{e}_i > 0 \\ \hat{e}_i = 0 & \text{if } \phi \sum a_{ij} \hat{e}_j \geq 1 \end{cases}$$

The first equation can be written as:

$$\hat{e}_i + \phi \sum a_{ij} \hat{e}_j = 1 \implies (\mathbf{I} + \phi \mathbf{A}_C) \mathbf{e}_C = \mathbf{1}$$

Note that the components of the sum $\sum a_{ij}\hat{e}_j$ are non-zero if and only if agent j is an active agent ($\hat{e}_j > 0$). Therefore the terms $a_{ij}\hat{e}_j$ represents the connection between active agents and can be written as $\mathbf{A}_C\mathbf{e}_C$. In the second part of the FOC, the case $\hat{e}_i = 0$, therefore a_{ij} represents the connections between an active and an inactive agent, therefore we can write:

$$\phi \sum a_{ij}\hat{e}_j \geq 1 \Leftrightarrow \phi \mathbf{A}_{N-C}\mathbf{e}_C \geq 1 \quad \blacksquare$$

In the next section, we discuss the conditions for interior and corner solutions in effort as characterized in Proposition 2 and derive necessary and sufficient conditions for uniqueness of Nash equilibria.

3 The potential function and uniqueness of equilibrium

We can study the uniqueness of Nash equilibria characterized in the previous section when we express the agent's maximization problem in terms of the potential function associated to the net payoff function U_i . In this section, we adopt techniques developed by BKD (2011) and apply them to our setting.

Specifically, consider a game where players take action $x_i \in X_i$ with $X = \prod_i X_i$ being the action space and payoffs are $V_i(x_i, \vec{x}_{-i})$. As in Monderer and Shapely (1996), a function $\Phi(x_i, \vec{x}_{-i})$ is called a *potential function* for this game if and only if, for all i :

$$\Phi(x_i, \vec{x}_{-i}) - \Phi(x'_i, \vec{x}_{-i}) = V_i(x_i, \vec{x}_{-i}) - V_i(x'_i, \vec{x}_{-i}) \quad \forall x_i, x'_i \in X_i, x_{-i} \in X_{-i}$$

Said differently, for $x \in \mathbb{R}$ and continuous, twice-differentiable payoffs V_i , there exist a potential function ($\frac{\partial V_i}{\partial x_i} = \frac{\partial \phi}{\partial x_i}$) if and only if:

$$\frac{\partial^2 V_i(\vec{x})}{\partial x_i \partial x_j} = \frac{\partial^2 V_i(\vec{x})}{\partial x_j \partial x_i}$$

Potential functions in strategic games were used for the first time by Rosenthal (1973). By explicitly constructing a potential function, he proved that for every game in a special class (called 'congestion games') there exists a pure-strategy equilibrium.⁹

Proposition 3. *The profile of effort levels, \mathbf{e} , is NE with best response function $f_i = \max\{0, 1 - \phi \sum_{j=1}^N a_{ij}e_j\}$ if and only if \mathbf{e} satisfy the Kuhn-Tucker conditions of potential function Φ . In other words, NE correspond to the local and global maxima and saddle points of Φ , where Φ is:*

$$\Phi(\mathbf{e}, G) = \left\{ \sum_{i=1}^N [e_i(1 - \pi) - \lambda e_i^2] - \frac{\gamma}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}e_i e_j \right\}$$

⁹A congestion game is a game where a group of identical agents have to choose their actions from a finite set of alternatives and agent payoffs depends on the number of agents choosing each alternative. Monderer and Shapely (1996) show that the class of congestion games coincides up to an isomorphism with the class of finite potential games (see BKD for more details).

Proof: We first provide the necessary and sufficient conditions that guarantee that Φ defined above is a proper potential function. Note that, by definition, Φ is a potential function for U_i if and only if $\frac{\partial U_i}{\partial e_i} = \frac{\partial \Phi}{\partial e_i}$ for all $i = 1, \dots, N$. By taking the derivative of Φ we have:

$$\begin{aligned}\frac{\partial \Phi}{\partial e_i} &= (1 - \pi) - 2\lambda e_i - \gamma \sum_{j=1}^N a_{ij} e_j = \frac{\partial U_i}{\partial e_i} \\ \frac{\partial \Phi}{\partial e_i} = 0 &\Rightarrow e_i^* = \frac{1 - \pi}{2\lambda} - \frac{\gamma}{2\lambda} \sum_{j=1}^N a_{ij} e_j^* = \beta - \phi \sum_{j=1}^N a_{ij} e_j^*\end{aligned}$$

Therefore the individual maximization problem can be re-written as the following constrained problem:

$$\max_{e_i} \Phi(\mathbf{e}, G) \quad \text{s.t. } e_i \geq 0 \text{ for all } i$$

Among the Kuhn-Tucker conditions associated with this problem are:

$$\begin{aligned}\frac{\partial \Phi}{\partial e_i} &= 0 \quad \text{and } e_i > 0 \\ \frac{\partial \Phi}{\partial e_i} &\leq 0 \quad \text{and } e_i = 0\end{aligned}$$

It is easy to see that these conditions correspond to the agent best response function exhibited earlier. Thus, the set of Nash Equilibria for any given network G is equivalent to the set of global and local maxima and saddle points of the potential function $\Phi(\mathbf{e}, G)$ on \mathbb{R}_+^n . ■

Next, re-write the potential function Φ using matrix notation:

$$\begin{aligned}\Phi(\mathbf{e}, G) &= \left\{ \sum_{i=1}^N [e_i(1 - \pi) - \lambda e_i^2] - \frac{\gamma}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} e_i e_j \right\} \\ &= \lambda \left\{ \sum_{i=1}^N \left[\left(\frac{1 - \pi}{\lambda} \right) e_i - e_i^2 \right] - \frac{\gamma}{2\lambda} \sum_{i=1}^N \sum_{j=1}^N a_{ij} e_i e_j \right\} \\ &= \lambda \left\{ \left(\frac{1 - \pi}{\lambda} \right) \mathbf{e}^T \mathbf{1} - \mathbf{e}^T (\mathbf{I} + \phi \mathbf{A}) \mathbf{e} \right\}\end{aligned}$$

This implies that the Hessian matrix of Φ is,

$$\nabla^2 \Phi = -\lambda (\mathbf{I} + \phi \mathbf{A})$$

The following result gives the conditions for uniqueness of Nash equilibrium in our setting.

Proposition 4. *There is a unique Nash Equilibrium if and only if $\phi < -\frac{1}{\alpha_{\min}(\mathbf{A})}$ where $\alpha_{\min}(\mathbf{A})$ is the minimum eigenvalue of \mathbf{A} .*

Proof: First, note that the potential function Φ has a unique maximum if and only if the matrix $(\mathbf{I} + \phi \mathbf{A})$ (the negative of Φ 's Hessian) is positive definite. Furthermore, BKD (2011)

show that the matrix $(\mathbf{I} + \phi\mathbf{A})$ is positive definite if and only if $\phi < -\frac{1}{\alpha_{\min}(\mathbf{A})}$, where $\alpha_{\min}(\mathbf{A})$ denotes the minimum eigenvalue of matrix $\mathbf{A} = [\frac{\lambda}{\gamma}(\mathbf{J} - \mathbf{I}) - \mathbf{G}]$. ■

So far we have shown that if $\phi < -\frac{1}{\alpha_{\min}(\mathbf{A})}$ there is a unique Nash equilibrium. This solution can be either interior (all $e_i > 0$) or corner (there are some $e_i = 0$). When $\phi > -\frac{1}{\alpha_{\min}(\mathbf{A})}$, the function Φ is non-concave and there might be multiple equilibria. It is easy to see that in this range of parameters, there is always a corner solution: because Φ is a non-concave function, there is a direction along which the potential function increases without bound, therefore there is at least one maximum point that is not interior. In addition, we know from Proposition 3 that any vector $\mathbf{e}(\phi, \mathbf{A})$ which globally maximizes Φ is a Nash equilibrium. Therefore, in this range of parameters there is always a Nash equilibrium which features a corner solution.

Lemma 2. *If $\phi > -\frac{1}{\alpha_{\min}(\mathbf{A})}$, there exists a Nash equilibrium which features corner solution (some i for which $\hat{e}_i = 0$).*

It is also interesting to note that it is never optimal for all agents to choose zero effort level in Nash equilibrium, hence the following result:

Proposition 5. *There is no Nash equilibrium in which $e_i = 0 \ \forall i$.*

Proof: By the first order conditions of the individual maximization problem we have,

$$\begin{aligned} e_i^* &= \left\{ \left(\frac{1-\pi}{2\lambda} \right) - \frac{\gamma}{2\lambda} \sum_{j \neq i} \left(\frac{\lambda}{\gamma} - g_{ij} \right) e_j \right\} \quad \text{if } e_i > 0 \\ e_i^* &= 0 \quad \text{if } \left(\frac{1-\pi}{2\lambda} \right) < \frac{\gamma}{2\lambda} \sum_{j \neq i} \left(\frac{\lambda}{\gamma} - g_{ij} \right) e_j \end{aligned} \quad (6)$$

If $e_j = 0 \ \forall j \neq i$, then agent i 's optimal effort level is $\frac{1-\pi}{2\lambda}$. Thus, the only way that agent i would choose zero effort is when there are some agent(s) whose effort level is larger than zero so that the right hand side of inequality (??) is larger than the left hand side. Therefore it is never optimal for all agents to choose zero effort level in equilibrium. ■

Another interesting result shown by BKD (2011) is the following property of *regular graphs*, that is networks in which every agent has the same number of connections:

Proposition 6. *For every regular graph, there is a unique interior Nash equilibrium if and only if $\phi < -\frac{1}{\lambda_{\min}(\mathbf{A})}$. If $\phi \geq -\frac{1}{\lambda_{\min}(\mathbf{A})}$, then both interior and corner solutions are present.*

Proof: see BKD (2011).

4 Properties of crime networks

In the previous sections we introduced the model and investigated the conditions for existence and uniqueness of interior and corner solutions. In this Section we present some interesting results when the cost of links $\delta = 0$ and also when $\delta \neq 0$.

Proposition 7. *Suppose $\delta = 0$. Then:*

(a) *The network maximizing aggregate crime, $E \equiv \sum_{i=1}^N e_i^*$ is the “full network”: the network where all pairs of players are connected.*

(b) *If, for given N , a network G' is obtained from G by adding more links among agents, i.e. $g_{ij} \leq g'_{ij} \forall i, j$ then $\sum_{i=1}^N e_i^*(G) \leq \sum_{i=1}^N e_i^*(G')$*

(c) *The maximal value of total equilibrium crime is increasing in the size of the network, N .*

Proof: (a) Note first that if G is the full network, with our assumptions, all e_i are equal to each other and positive. Look at the optimality conditions, (??). We will show that among all networks G between active players (i.e. such that $e_i > 0$) the full network maximizes total crime. The i^{th} player's first order condition can be written as:

$$\beta_1 = 2\phi_1 e_i + \sum_{j \neq i} (\phi_1 - 1_{g_{ij}=1}) e_j$$

where $1_{g_{ij}=1}$ is the indicator function taking value of 1 if $g_{ij} = 1$ and zero otherwise. Notice that:

$$\begin{aligned} 2\phi_1 e_i + \sum_{j \neq i} (\phi_1 - 1_{g_{ij}=1}) e_j &\geq (\phi_1 + 1) e_i + \sum_{k=1}^N (\phi_1 - 1) e_k \\ \Rightarrow \beta_1 &\geq (\phi_1 + 1) e_i + (\phi_1 - 1) E(G) \end{aligned}$$

with equality only if all g_{ij} for $j \neq i$ are ones (i.e. G is the full network). Adding up we obtain:

$$\begin{aligned} N\beta_1 &\geq [\phi_1 + 1 + N(\phi_1 - 1)] E(G) \\ \Rightarrow E(G) &\leq \frac{\beta_1}{\frac{\phi_1+1}{N} + \phi_1 - 1} \\ \Rightarrow E(G) &\leq \frac{1 - \pi}{\frac{\lambda+\gamma}{N} + (\lambda - \gamma)} \end{aligned}$$

for all networks G and with equality only if G is the full network. Thus total crime, E is maximized when G is the full network as ϕ_1, β_1 and N do not depend on G .

(b) This result follows easily as in BCZ, Proposition 2.

(c) From (a) we know that for the full network, $E = \frac{\beta_1}{\frac{\phi_1+1}{N} + \phi_1 - 1}$ which is increasing in N since $\phi_1 > 0$. ■

The result in part (c) together with that in (a) imply that if N is bounded from above, then maximum crime is achieved for the full network of maximum possible size. Of course, this may be too costly and so the optimal (profit maximizing, rather than crime maximizing) network can be of small size or be different than the full network (see next section). Also, the individual equilibrium effort for the full network with N players if $\delta = 0$ is $e^{full}(N) = \frac{\beta}{\phi_1+1+N(\phi_1-1)}$ which is decreasing in N if $\phi_1 > 1$ (i.e. if $\lambda\pi > \gamma$ – marginal effort costs are larger than marginal effort benefits) and increasing in N otherwise.

Breaking from the matrix notation we can write the FOCs of the model in an N-player

environment as expressing the best-response effort level of each player, e_i at an interior solution:

$$e_i = \left[\left(\frac{1-\pi}{2\lambda} \right) - \left(\frac{\delta}{2\lambda} \sum_{j=1}^N g_{ij} \right) + \left(\frac{\gamma}{2\lambda} \sum_{j=1}^N g_{ij} e_j \right) - \left(\frac{1}{2} \sum_{j \neq i}^N e_j \right) \right]$$

Re-arranging in this way shows that the first term, $(\frac{1-\pi}{2\lambda})$, is the standalone or “autarky” level of effort. It is as if the individual was without any connection to others. The second term $\frac{\delta}{2\lambda} \sum_{j=1}^N g_{ij}$ can be broken into two parts: those that are directly linked to actor i and those that are not directly linked to i . In the former case we have $g_{ij} = 1$, in the latter case, $g_{ij} = 0$. Consequently, the summation can be written as:

$$\frac{\delta}{2\lambda} \sum_{j=1}^N g_{ij} = \frac{\delta}{2\lambda} (l_i)$$

where l_i is the number of direct links that player i has with other players in the network. Rewriting:

$$e_i = \left[\left(\frac{1-\pi}{2\lambda} \right) - \left(\frac{\delta}{2\lambda} (l_i) \right) + \left(\frac{\gamma}{2\lambda} \sum_{j=1}^N g_{ij} e_j \right) - \left(\frac{1}{2} \sum_{j \neq i}^N e_j \right) \right].$$

If we now divide the third term into its linked and disconnected segments where the expression $L(i)$ refers to the direct links that agent i has with others and the 0 reflects the observation that each $g_{ij} = 0$ if there is no link, we have:

$$\frac{\gamma}{2\lambda} \sum_{L(i)} g_{ij} e_j + 0,$$

so that we can also express the solution as:

$$e_i = \left[\left(\frac{1-\pi-\delta l_i}{2\lambda} \right) + \left(\frac{\gamma}{2\lambda} \sum_{L(i)} g_{ij} e_j \right) - \left(\frac{1}{2} \sum_{j \neq i}^N e_j \right) \right] \quad (7)$$

and re-arrange so that:

$$e_i = \left[e^{autarky} + \frac{\gamma}{2\lambda} \sum_{L(i)} g_{ij} e_j - \frac{\delta l_i}{2\lambda} - \frac{1}{2} \sum_{j \neq i}^N e_j \right]$$

In this expression the autarky level of effort, $e^{autarky} \equiv \frac{1-\pi}{2\lambda}$ is augmented by the increase in output associated with the links to others $\frac{\gamma}{2\lambda} \sum_{L(i)} g_{ij} e_j$. We see that reducing the benefits from being in the network is the cost of the number of links, $\frac{\delta l_i}{2\lambda}$ and the increased likelihood of being caught which is proportional to the sum of the effort levels of all the other members of the network, $\frac{1}{2} \sum_{j \neq i}^N e_j$. This is the structure from which it is convenient to calculate the equilibrium effort levels in the networks and it is used in Appendix 2 to provide an example of

a solution for a four-player network. The algebraic characterization of the equilibrium levels of effort for an interior solution leads to two more propositions:

Proposition 8. *Each player's equilibrium payoff, U_i^* is quadratic in his effort, e_i^* .*

Proof: From the earlier expression for the payoff of each player, U_i^* , we have:

$$\begin{aligned}
U_i^* &= e_i^* \left[(1 + \gamma \sum_{j=1}^N g_{ij} e_j^*) - (\pi + \lambda \sum_{j=1}^N e_j^* + \delta \sum_{j=1}^N g_{ij}) \right] \\
&= e_i^* \left[(1 - \pi) + \gamma \sum_{j=1}^N g_{ij} e_j^* - \lambda \sum_{j=1}^N e_j^* - \delta \sum_{j=1}^N g_{ij} \right] \\
&= e_i^* \left[(1 - \pi) + \sum_{j=1}^N (\gamma g_{ij} - \lambda) e_j^* - \delta l_i \right] \\
&= e_i^* \left[(1 - \pi) - \delta l_i + (\gamma - \lambda) \sum_{j \in L(i)} e_j^* - \lambda \sum_{j \notin L(i)} e_j^* - \lambda e_i^* \right]
\end{aligned}$$

On the other hand we know that at an interior solution (from the FOCs):

$$\begin{aligned}
e_i^* &= \frac{1 - \pi}{2\lambda} - \frac{\delta l_i}{2\lambda} - \frac{1}{2} \sum_{j \neq i} e_j^* + \frac{\gamma}{2\lambda} \sum_{j \in L(i)} e_j^* \\
&= \frac{1}{2\lambda} \left[1 - \pi - \delta l_i + (\gamma - \lambda) \sum_{j \in L(i)} e_j^* - \lambda \sum_{j \notin L(i)} e_j^* \right]. \tag{8}
\end{aligned}$$

Combine the final expression for profit and the last expression for e_i^* and we observe that:

$$U_i^* = \lambda (e_i^*)^2 \quad \forall i = 1, \dots, N$$

Thus, if the solutions are interior, profits are always non-negative and proportional to the square of own effort in equilibrium. If the solution is a corner ($e_i^* = 0$) we trivially have $U_i^* = 0$ as well. ■

An interesting question that arises here is about the relationship between the equilibrium effort level and the position of agents in the network. A close look at the characterization of the equilibrium level of effort in (??) leads to the following result:

Proposition 9. *The “Lazy colleague” theorem*

- (a) *If $\delta = 0$, an inactive agent ($e_i^* = 0$) is connected to agents with lower efforts in total than an active agent.*
- (b) *If the sets of neighbors of two active agents are nested, the agent with more active neighbors chooses higher effort level in equilibrium.*
- (c) *If $\delta \neq 0$ and two active agents have the same set of active neighbors, then the agent with more inactive neighbors chooses a lower effort level in equilibrium.*

Proof: a) Suppose $e_i^* = 0$ for some $i = 1, \dots, N$ and $e_j^* > 0$ for the rest. Then the FOCs for an interior Nash equilibrium in efforts are:

$$\begin{aligned} \frac{1 - \pi}{\lambda} - \frac{\delta l_i}{\lambda} - \sum_{k=1}^n e_k^* + \frac{\gamma}{\lambda} \sum_{k \in L(i)} e_k^* &< 0 \quad \text{for player } i \text{ and,} \\ e_j^* = \frac{1 - \pi}{\lambda} - \frac{\delta l_i}{\lambda} - \sum_{k=1}^n e_k^* + \frac{\gamma}{\lambda} \sum_{k \in L(j)} e_k^* &\quad \text{for any player } j. \end{aligned}$$

Suppose $\delta = 0$. Then it must be that $\sum_{k \in L(i)} e_k^* < \sum_{k \in L(j)} e_k^*$. This suggests that if there are no costs to links, then for an individual who gives no effort, his links will be to those who have lower efforts in total than the players who are linked to someone with a positive effort. This might be thought of as the ‘‘Lazy Colleague’’ theorem. Of course if $\delta \neq 0$, it still may be the case that those in association with colleagues who provide no effort will provide less effort so long as they have sufficiently many costly connections, or if the productive colleague has sufficiently few connections. These are summarized by the necessary condition:

$$\frac{\delta (l_j - l_i)}{\gamma} + \sum_{k \in L(i)} e_k^* < \sum_{k \in L(j)} e_k^*$$

b) By comparing two active agents whose neighbors are nested sets we see that for $\phi < 1$ the agents with the larger set of active neighbors play more, i.e. it is optimal for the agent who is connected to more outside sources to take higher action. To see this let N_i^C, N_j^C be the set of agent i, j 's active neighbors, and let l_i^C, l_j^C be the number of agent i, j 's active neighbors (active degree) such that:

$$\begin{aligned} e_i^* &= \frac{1 - \pi}{\lambda} + \frac{\gamma}{\lambda} \sum_{j \in N_i^C} e_j^* - \sum_{j=1}^N e_j^* \\ N_k^C \subseteq N_i^C &\Rightarrow l_k^C \leq l_i^C \\ \sum_{j \in N_i^C} e_j^* &> \sum_{j \in N_k^C} e_j^* \Rightarrow e_i^* > e_k^* \end{aligned}$$

Where l_i^C is the active degree of the number of active neighbors of agent i .

c) And Finally, if we assume that the cost of link formation $\delta \neq 0$, then the above equation would become:

$$e_i^* = \frac{1 - \pi}{\lambda} + \frac{\gamma}{\lambda} \sum_{j \in N_i^C} e_j^* - \sum_{j=1}^N e_j^* - \frac{\delta}{\lambda} l_i$$

Where l_i is the total number of agent i 's neighbors. Now assume that agent i, j have the exact same active friends, but agent i has more inactive friends. It is easy to see that:

$$\text{If } l_i^C = l_j^C \text{ and } l_i > l_j \Rightarrow e_i^* < e_j^*$$

This suggests that when an active agent is connected to inactive agents he puts less effort into the criminal activity. Being friend with an active agent may have positive or negative effects. As long as $\phi e_k^* - \delta/\lambda$ is positive, befriending that active agent will increase agent's i 's effort level. But a less active agent (i.e. $\phi e_k^* - \delta/\lambda < 0$) can decrease agent i 's effort level. ■

4.1 A Planner's Problem

So far we have studied the strategic decision making by each individual. However, a series of interesting results arise when we contrast these results by the solutions to a planner's problem. In the reminder of this section, we will look at the effort choice problem from a planner's (criminal boss) perspective and characterize the optimal solution assuming this planner maximizes the overall profit of the criminal network.

The aggregate net payoff of all the agents in criminal network G is given by:

$$\begin{aligned} U(\mathbf{e}, G) &= \sum_{i=1}^N U_i(\mathbf{e}, G) = \\ &= \left\{ \sum_{i=1}^N e_i(1 - \pi) - \gamma \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij} \right) e_i e_j \right\} \end{aligned}$$

The first order conditions of maximizing the above with respect to e_i imply (assuming interior solution):

$$\tilde{e}_i = \frac{1 - \pi}{2\lambda} - \frac{\gamma}{\lambda} \sum_{j \neq i} \left(\frac{\lambda}{\gamma} - g_{ij} \right) \tilde{e}_j \quad (9)$$

We contrast the above expression with the effort level that maximizes individual payoff given $e_j \forall j \neq i$ and a given network G (again, assuming interior solution),

$$e_i^* = \frac{1 - \pi}{2\lambda} - \frac{\gamma}{2\lambda} \sum_{j \neq i} \left(\frac{\lambda}{\gamma} - g_{ij} \right) e_j^*.$$

The first result in this section characterizes the relationship between the total effort level and total payoff in the planner's problem:

Proposition 10. *Total payoff is proportional to total effort level in the planner's solution.*

Proof: From the first order conditions we have:

$$\begin{aligned} (1 - \pi) - 2\gamma \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij} \right) \tilde{e}_j &= 0 \\ \Rightarrow \gamma \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij} \right) \tilde{e}_j &= \frac{1 - \pi}{2} \end{aligned}$$

By substituting the above results in the total payoff function we obtain:

$$\begin{aligned}\tilde{U}(\mathbf{e}, G) &= \left\{ \sum_{i=1}^N \tilde{e}_i(1 - \pi) - \gamma \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij}\right) \tilde{e}_i \tilde{e}_j \right\} \\ \Rightarrow \tilde{U}(\mathbf{e}, G) &= \left\{ \sum_{i=1}^N \tilde{e}_i(1 - \pi) - \sum_{i=1}^N \left(\frac{1 - \pi}{2}\right) \tilde{e}_i \right\} = \sum_{i=1}^N \left(\frac{1 - \pi}{2}\right) \tilde{e}_i \\ \text{Let } \tilde{E}(G) &= \sum_{i=1}^N \tilde{e}_i \quad \Rightarrow \quad \tilde{U}(\mathbf{e}, G) = \left(\frac{1 - \pi}{2}\right) \tilde{E}(G)\end{aligned}$$

Q.E.D. ■

We next revisit the results from proposition (??) in the planner's problem.

Proposition 11. *The full network maximizes both total effort and total payoff in the planner's problem. The optimal values of total effort level and total payoff are both increasing in the number of agents in the network.*

Proof: From the first order conditions we have:

$$\begin{aligned}\frac{\gamma}{\lambda} \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij}\right) \tilde{e}_j &= \frac{1 - \pi}{2\lambda} \\ \Rightarrow \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij}\right) \tilde{e}_j &\geq \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - 1\right) \tilde{e}_j + \tilde{e}_i,\end{aligned}$$

with equality if G is a full network, i.e. $g_{ij} = 1$. In a full network, because of symmetry we have $\tilde{e}_i = \tilde{e} \forall i$. Adding up both sides we get:

$$\begin{aligned}\sum_{i=1}^N \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij}\right) \tilde{e}_j &\geq \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - 1\right) \tilde{e} + \sum_{i=1}^N \tilde{e} \\ \Rightarrow \frac{N(1 - \pi)}{2\gamma} &\geq \left(1 + N \frac{\lambda - \gamma}{\gamma}\right) \tilde{E}(G) \\ \Rightarrow \tilde{E}(G) &\leq \frac{N(1 - \pi)}{2(\gamma + N(\lambda - \gamma))} = \frac{(1 - \pi)}{2\left(\frac{\gamma}{N} + (\lambda - \gamma)\right)},\end{aligned}$$

with equality if the network is full. Therefore, from the previous Proposition,

$$\tilde{U}(\mathbf{e}, G) \leq \frac{N(1 - \pi)^2}{4(\gamma + N(\lambda - \gamma))} = \frac{(1 - \pi)^2}{4\left(\frac{\gamma}{N} + (\lambda - \gamma)\right)}$$

It is clear from these results that both total effort \tilde{E} and total payoff \tilde{U} in the planner's problem are maximized when G is the full network. Also note that these values are increasing as the number of agents N in the network increases. However, the optimal value of the individual

effort level is decreasing as the number of agents in the full network increases:

$$\tilde{e}_{full} = \frac{(1 - \pi)}{2(\gamma + N(\lambda - \gamma))}$$

■

In the following example, we further contrast the results for the maximization problem from an individual and planner's perspectives. We compare the total payoff when individuals choose their own effort levels given a specific network to the maximized total payoff when a planner maximizes the overall payoff. We consider the results for two extreme cases: empty and full networks. The following example summarizes these results:

Example – comparing the Nash and planner's solutions

As we have seen before, the utility function for an individual in a criminal network G and his optimal effort level are given by:

$$u_i(\mathbf{e}, G) = \left\{ e_i(1 - \pi) - \gamma \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij} \right) e_i e_j \right\}$$

$$e_i^* = \frac{1 - \pi}{2\lambda} - \frac{\gamma}{2\lambda} \sum_{j \neq i} \left(\frac{\lambda}{\gamma} - g_{ij} \right) e_j^*$$

On the other hand, the total utility and the first order conditions of the planner's problem are:

$$U(\mathbf{e}, G) = \left\{ \sum_{i=1}^N e_i(1 - \pi) - \gamma \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\lambda}{\gamma} - g_{ij} \right) e_i e_j \right\}$$

$$\tilde{e}_i = \frac{1 - \pi}{2\lambda} - \frac{\gamma}{\lambda} \sum_{j \neq i} \left(\frac{\lambda}{\gamma} - g_{ij} \right) \tilde{e}_j$$

Therefore the optimal effort level for the empty and full networks from both individual and planner's perspective are:

- **Empty network**

1. Individual's problem:

$$e_{i,\text{empty}}^* = \frac{1 - \pi}{2\lambda} - \frac{1}{2} \sum_{j \neq i} e_{j,\text{empty}}^* = \frac{1 - \pi}{(N + 1)\lambda}$$

$$u_i^*(\vec{e}, \mathbf{G}) = \frac{(1 - \pi)^2}{(N + 1)\lambda} - \lambda N \frac{(1 - \pi)^2}{(N + 1)^2 \lambda^2} = \frac{(1 - \pi)^2}{(N + 1)^2 \lambda}$$

$$\Rightarrow U^* = \frac{N(1 - \pi)^2}{(N + 1)^2 \lambda}$$

2. Planner's problem:

$$\begin{aligned}\tilde{e}_i &= \frac{1-\pi}{2\lambda} - \sum_{j \neq i} \tilde{e}_j \quad \& \quad \tilde{e}_i = \tilde{e}_j = \tilde{e} \\ \Rightarrow \tilde{e} &= \frac{1-\pi}{2N\lambda} \quad \Rightarrow \quad \tilde{U}_{empty} = \frac{(1-\pi)^2}{4\lambda}\end{aligned}$$

Comparing these results it is easy to see that $e_{i,empty}^* > \tilde{e}_{i,empty}$ and $U_{empty}^* < \tilde{U}_{empty}$ if and only if $N > 1$. Therefore by comparing the results from individual decisions with the planner's problem, we see that, as expected, individuals put higher effort into the criminal activity, but this results in a lower overall utility than the first best.

• **Full Network:**

1. Individual's problem:

$$\begin{aligned}e_{full}^* &= \frac{1-\pi}{(\lambda + \gamma + N(\lambda - \gamma))} \\ u_i^*(\vec{e}, \mathbf{G}) &= \frac{\lambda(1-\pi)^2}{(\lambda + \gamma + N(\lambda - \gamma))^2} \\ \Rightarrow U_{full}^* &= \frac{N\lambda(1-\pi)^2}{(\lambda + \gamma + N(\lambda - \gamma))^2}\end{aligned}$$

2. Planner's problem:

$$\begin{aligned}\tilde{e}_{full} &= \frac{(1-\pi)}{2(\gamma + N(\lambda - \gamma))} \\ \Rightarrow \tilde{U}_{full} &= \frac{N(1-\pi)^2}{4(\gamma + N(\lambda - \gamma))} = \frac{(1-\pi)^2}{4(\frac{\gamma}{N} + (\lambda - \gamma))}\end{aligned}$$

It is easy to show that $e_{full}^* > \tilde{e}_{full}$ if and only if $\lambda \geq \gamma$, and from lemma (1) we know that $\lambda \geq \gamma$ is the necessary and sufficient condition for existence of equilibrium. Also it is easy to see that $U_{full}^* < \tilde{U}_{full}$.

5 Optimal networks and crime-deterrent policies (incomplete)

Given the equilibrium individual crime levels derived above we can compute the overall equilibrium crime level, $E = \sum_{i=1}^N e_i$, as well as the overall equilibrium profit (surplus) level, $\Pi = \sum_{i=1}^N (y_i - c_i)$ for each possible network structure G . These two aggregates play an important role in the subsequent analysis. On the one hand they are informative about the type of

networks we would expect to observe if agents are optimizing according to our model. A standard competition or group selection argument suggests that the network that maximizes total surplus, Π , is likely to be the optimal network in the long run. The same result can be obtained if we assume that a “planner” (boss or Godfather) designs the network to extract maximum surplus. On the other hand, the level of total crime that is likely to emerge optimally may be guiding the design and implementation of most effective crime-reducing policies that an outside authority (e.g. the police) would like to implement. We will study the interaction between the optimal network structure in a particular policy environment (including lack of policing) and the best policies from a list of several possibilities given the resulting optimal network.

The optimal network structure will in general reflect the policy environment in which the network operates. This is important and arises as an application of the standard Lucas critique (Lucas, 1976). Policy design must recognize that the actor’s response may be a function of the policies that are employed. Moreover, the optimal network structure may change (potentially at a cost) as a given policy is implemented. Although it applies to any policy, consider the policy of “removing the key player” proposed in BCZ (2004). BCZ take a given network G_0 and identify the “key player”: the agent whose removal from the network results in the largest drop in the overall crime level. There are two potential weaknesses of this approach as a guide to policy effectiveness. First, when determining who the “key player” is, we assume that after this player is removed, the remaining network does not re-optimize.¹⁰ Second, and more importantly, the initial network is not necessarily the optimal network that would result were the criminals to recognize that a particular policy is in place.¹¹

Practically, since analytical results cannot be derived easily, to find the optimal network for a given policy environment we need to search over all possible networks, compute the expected profits for each and choose the one with the highest value. In principle, for a given N and our bidirectional link structure, all possible networks are $2^{N(N-1)/2}$ - a number that becomes huge even for very low N . However many of these networks are equivalent, or in graph-theory language, isomorphic. They can be obtained from each other by simply re-labeling the nodes. Obviously, in our setting all isomorphic networks will yield the same crime and profit levels so we only search over non-isomorphic networks when finding the optimal one. We achieve this by using computed data made publicly available by Brendan McKay from the Australian National University. This data set provides a list of the adjacency matrices (in a special compressed format, see Appendix 1) of all non-isomorphic graphs for $N = 2, 3, \dots, 9$. The outcomes of the optimization for profit, effort and network structure depend on the parameter values for the set of λ, π, δ and γ . We use our theoretical results as a guideline to choose appropriate values for these values that guarantee the existence of solutions. And we perform our analysis using 10 values for each one of these parameters so that we assess 10000 parameter combinations.

As an illustration, we are going to examine several crime-combatting policies such as re-

¹⁰In principle, of course, links can be re-arranged to maximize profits so that the residual network is not simply the original network set of links less one player. The key player approach in BCZ assumes that no such re-arrangement of the network occurs. However, when it is possible to re-arrange links, the optimal network may be a different one.

¹¹In BCZ (2004) of course the analysis is conducted with respect to a given network and is in no sense an optimal network.

moving the key player, removing a random player, removing a link, etc. We will allow these events to happen with a certain probability (a parameter) and look at (i) the case in which after the removal the network re-optimizes (potentially at a cost), and (ii) the case in which the network does not re-optimize. The optimality criterion will be the ex-ante expected profit obtained given the policy. Knowing the optimal network structures that arise for a given policy means that we can study the relative effectiveness of various policies that are defined as the reduction in the aggregate crime effort.

6 Discussion and conclusions

We develop a model that optimizes overall profits in a criminal network by varying both individual crime effort levels and the network configuration. We characterized conditions for existence and uniqueness of corner and interior Nash equilibria in efforts and gave a characterization of some properties of the solution, including individual and aggregate crime effort and total profits.

The next immediate steps are to examine several policy environments using the McKay data for a range of parameters. This analysis is important because the policy environment affects the formation of the optimal network and as discussed earlier policy design should take the agents' responses to their implementation into account. This highlights the need for a more detailed understanding of the impact of the policy environment on the network optimization problem.

Finally, there are a variety of challenges to this methodology and a number of interesting questions to be posed. Many networks are larger than ten or eleven players. Can we deal with larger numbers in a systematic way that still preserve the spirit of optimization? If we know the observed structure of a crime network, how is that information to be integrated? Can it be used to reduce the number of networks over which we need to search? Can it help us identify the relevant parameters and their magnitudes? There are reasonable questions about what procedure is relevant when a network is stressed. Should the removal of a player simply mean that the network continues with one fewer members but leave links intact? Or should new links be forged without adding an additional player? Does the network learn? Among the questions that we can address are those related to knowledge about the network based on incomplete information about observed nodes. Can we say something about the size or structure of the network by observing one node? How much can we learn about the network knowing the links of one player to another? Is it the case that certain structures are favored in real world environments? And many more.

References

- [1] Aitken, C., Moore, D., Higgs, P., Kelsall, J. and Kerger, M. (2002), 'The Impact of a Police Crackdown on a Street Drug Scene: Evidence from the Street', *International Journal of Drug Policy*, 13: 193–202.

- [2] Bala, V and Goyal, S. "A Noncooperative Model of Network Formation" *Econometrica* 68 (2000): 1181-1229.
- [3] Ballester, C., A. Calvo-Armengol, and Y. Zenou, "Who's Who in Crime Networks. Wanted: the Key Player" working paper, (2004).
- [4] Bramoulle, Yann, Rachel Kranton, and Martin D'Amours (2011), "Strategic interaction and networks.", working paper.
- [5] Bruinsma, G. and W. Bernasco, "Criminal groups and transnational illegal markets: A more detailed examination on the basis of social network theory", *Crime, Law and Social Change: An Interdisciplinary Journal* 41 (2004): 79-94.
- [6] Calvo-Armengol, A. and Y. Zenou "Social Networks and Crime Decisions: The Role of Social Structure in Facilitating Delinquent Behavior" *International Economic Review* 45 (2004): 939-58
- [7] Carley, K., J-S .Lee, and D. Krackhardt, "Destabilizing Networks", *Connections* 24 (2002): 79-92.
- [8] Carrington, Peter J., Scott, John and Wasserman, Stanley. *Models and Methods in Social Network Analysis* (New York: Cambridge University Press, 2005): 1.
- [9] Cherbonneau, M. and Copes, H. 'Drive it like you Stole it' Auto Theft and the Illusion of Normalcy" *The British Journal of Criminology* 46:193-211 (2006)
- [10] Coles, Nigel, "It's Not What You Know—It's Who You Know That Counts: Analysing Serious Crime Groups as Social Networks" *British Journal of Criminology* 41 (2001): 580-594.
- [11] Easton, S. and Karaivanov, A. "Understanding Optimal Criminal Networks" *Global Crime* 10 (2009): 41-69.
- [12] Frank, O. "Statistical Estimation of Co-Offending Youth Networks" *Social Networks* 23 (2001): 203-214
- [13] Granovetter, M. "The Strength of Weak Ties," *American Journal of Sociology* 78 (1973): 1360-80.
- [14] Hoffer, L. *Junkie business: The evolution and operation of a heroin dealing network*. (Ann Arbor, MI: UMI Press, 2002)
- [15] Hughes, D. "The" Natasha" trade: The transnational shadow market of trafficking in women" *Journal of International Affairs*, Spring 2000 Vol. 53 Issue 2, p625, 27p.
- [16] Jackson, M. and Wolinski, A. "A strategic Model for Economic and Social Networks" *Journal of Economic Theory* 71 (1996) 44-74.

- [17] Jackson, M., “The Stability and Efficiency of Economic and Social Networks,” in S. Koray and M. Sertel (eds.), *Advances in Economic Design* (Springer-Verlag: Heidelberg 2003)
- [18] Jackson, M., “A Survey of Models of Network Formation: Stability and Efficiency,” in G. Demange and M. Wooders (eds.) *Group Formation in Economics: Networks, Clubs and Coalitions* (Cambridge University Press: Cambridge U.K., 2004)
- [19] Jackson, M., “The Economics of Social Networks”, ch. 1 in *Advances in Economics and Econometrics, Theory and Applications*, R. Blundell, W. Newey, and T. Persson, (eds.), (Cambridge University Press, 2006).
- [20] Jackson, M., “The Study of Social Networks in Economics”, in *The Missing Links: Formation and Decay of Economic Networks*, J. Rauch (ed.), (Russell Sage Foundation, 2007);
- [21] Jackson, M., “Network Formation”, in *The New Palgrave Dictionary of Economics and the Law*, (MacMillan Press, 2008).
- [22] Jacobs, B. (1999), *Dealing Crack: The Social World of Streetcorner Selling*. Boston, MA: Northeastern University Press.
- [23] Johnson, B. and Natarajan, M. (1995), ‘Strategies to Avoid Arrest: Crack Sellers’ Response to Intensified Policing’, *American Journal of Police*, 14: 49–69.
- [24] Krebs, V. ”Mapping Networks of Terrorist Cells” *Connections* 24(3):43-52, 2002.
- [25] Lucas, R.E. Jr. ”Econometric Policy Evaluation: A Critique,” in Karl Brunner and Allan Meltzer eds., *The Phillips Curve and Labor Markets*, Vol. 1 The Carnegie-Rochester Conferences on Public Policy, *Journal of Monetary Economics* Suppl. (1976): 19-46.
- [26] Manning, P. (1978), ‘Lying, Secrecy and Social Change’, in P. Manning, and J. Van Maanen, eds, *Policing: A View From the Street*, 238–55. New York: Random House.
- [27] McKay, B. “Practical Graph Isomorphism” *Congressus Numerantium* v. 30 (1981): 45-87.
- [28] Monderer, Dov, and Lloyd S. Shapley. . “Potential games ” *Games and economic behavior* 14.1 (1996): 124-143.
- [29] Moreno, J., *Who shall survive?: A new approach to the problem of human interrelations* (Nervous and mental disease publishing Co.: Washington, D.C. 1934)
- [30] North, D., “The Evolution of Efficient Markets in History”, in *Capitalism in Context: Essays on Economic Development and Cultural Change in Honor of R.M. Hartwell*, University of Chicago Press, 1994.
- [31] Raab, J. and H. Milward, , ”Dark Networks as Problems” *Journal of Public Administration Research and Theory* (2003): 413-439
- [32] Rosenthal, R. W. “A Class of Games Possessing Pure-Strategy Nash Equilibria”, in *Int. J. Game Theory* 2 (1973): 6567.

- [33] Sarnecki J. (2001) *Delinquent networks: Youth co-offending in Stockholm*, Cambridge, UK: Cambridge University Press
- [34] Zimmer, L. (1990), ‘Proactive Policing Against Street-Level Drug Trafficking’, *American Journal of Police*, 9: 43–75.

7 Appendix 1 – Enumerating all non-isomorphic networks of given size

One of the main contributions of this paper is that we study “optimal networks”, i.e. networks that maximize some economically relevant criterion among *all possible networks* of a given size. In principle, there are $2^{N(N-1)/2}$ networks of size N . Even at $N = 7$ these are already 2^{21} i.e. more than 2 million networks, thus if we were to compute an equilibrium for each of them the computational time required will grow exponentially with network size. On the other hand, it is clear that many of the possible networks are effectively the same modulo some permutation of the numbering of vertices. For example, for $N = 7$ there are only 1044 unique networks. Thus, for our purposes and to avoid costly duplication of time and effort we only need to compute the equilibria for the different, or as they are known in graph theory, *non-isomorphic* networks. As mentioned in the introduction, it turns out that generating all non-isomorphic graphs of a given size is a hard problem in graph theory and computer science, one that has not been solved for any N .

Fortunately for us, Brendan McKay from the Department of Computer Science at the Australian National University, has developed an algorithm to compute all non-isomorphic networks for up to $N = 12$ and has made the data, i.e. the adjacency matrices of those networks publicly available on his website. These adjacency matrices data come in a special format designed by Prof. McKay (the “g6 format”) which minimizes the amount of storage necessary. The algorithm stores the upper diagonal part of the adjacency matrices as a string of ASCII symbols. Below we explain how one can convert a .g6 file into the set of its corresponding adjacency matrices. A Matlab file performing the conversion is available upon request by the authors.

Matab Algorithm for Converting the McKay data into adjacency matrices

1. Open a .g6 file provided by Prof. McKay at <http://cs.anu.edu.au/people/bdm/data/graphs.html>. A string of ASCII codes is returned, separated by “linefeed”, ASCII=10 (use `fopen` to open the file and `fread` to read its contents).
2. Find the number of lines (number of linefeed symbols) - this corresponds to the number of networks in the file, N .
3. Eliminate the linefeed symbols and reshape the remaining ASCII symbols into a matrix with N rows each corresponding to a non-isomorphic network.
4. Subtract 63 from the ASCII codes of the elements of the matrix in 3.
5. For each row in the resulting matrix from 4 (i.e. each network) perform the following:
 - (a) find the number of vertices, element 1 of each row
 - (b) convert the rest of the row elements from a decimal number into groups of 6-digit binary numbers (the g6 format uses only 6 bits).

(c) the result from (b) which is a sequence of zeros and ones is the upper diagonal part of the adjacency matrix of the current network, going column by column, i.e. starting at element (1,2), then (1,3), ..(1,N), (2,3), etc.

8 Appendix 2 – Solving the case $N = 4$

In this appendix we solve explicitly for the effort levels in a four player system. Recall from the model section in the text, some of which is repeated here, that the initial benefits to crime are associated with the maximization of "utility" or profit arising from income and costs associated with the network described below:

$$U_i(G, \mathbf{e}) = y_i(G, \mathbf{e}) - c_i(G, \mathbf{e})$$

where:

$$y_i(G, \mathbf{e}) = e_i \left(1 + \gamma \sum_{j=1}^N g_{ij} e_j \right)$$

and

$$c_i(G, \mathbf{e}) = e_i \left(\pi + \lambda \sum_{j=1}^N e_j + \delta \sum_{j=1}^N g_{ij} \right)$$

Differentiating:

$$\frac{\partial y_i}{\partial e_i} = 1 + \gamma \sum_{j=1}^N g_{ij} e_j$$

and

$$\frac{\partial c_i}{\partial e_i} = \pi + \lambda \sum_{j \neq i}^N e_j + 2\lambda e_i + \delta \sum_{j=1}^N g_{ij}$$

So that

$$\frac{\partial U_i}{\partial e_i} = \frac{\partial y_i}{\partial e_i} - \frac{\partial c_i}{\partial e_i} = \left(1 + \gamma \sum_{j=1}^N g_{ij} e_j \right) - \left(\pi + \lambda \sum_{j \neq i}^N e_j + 2\lambda e_i + \delta \sum_{j=1}^N g_{ij} \right) = 0$$

which solves for e_i :

$$e_i = \left[\left(\frac{1 - \pi}{2\lambda} \right) - \left(\frac{\delta}{2\lambda} \sum_{j=1}^N g_{ij} \right) + \left(\frac{\gamma}{2\lambda} \sum_{j=1}^N g_{ij} e_j \right) - \left(\frac{1}{2} \sum_{j \neq i}^N e_j \right) \right]$$

Following the discussion in the text, we can rewrite this as:

$$e_i = \left[e^{autarky} - \left(\frac{\delta (\#L(i))}{2\lambda} \right) + \left(\frac{\gamma}{2\lambda} \sum_{L(i)} g_{ij} e_j \right) - \left(\frac{1}{2} \sum_{j \neq i}^N e_j \right) \right]$$

This is the structure from which it is convenient to calculate the equilibrium effort levels in the networks. An explicit example is discussed below.

For the four player network there are six possible non-isomorphic configurations: the star, the wheel, the line, the kite (or cell), the full and the half-full networks. For the star, the first effort level is associated with the player (1) in the centre of the star. The effort level depends on the autarky level of effort, $\frac{1-\pi}{2\lambda}$, less the cost of 3 links, $\frac{\delta}{2\lambda}(3)$, one to each of the points of the star; the value of the connections the players 2,3, and 4, $\frac{\gamma}{2\lambda}(e_2 + e_3 + e_4)$, and the cost of being involved with the people in the network, $\frac{1}{2}(e_2 + e_3 + e_4)$. In the case of the other three players, each of whom is at a point of the star, we again have the autarky level, $\frac{1-\pi}{2\lambda}$, from which is subtracted the cost of the single connection to the centre, $\frac{\delta}{2\lambda}(1)$, the value of the connection to the centre player, player 1, $\frac{\gamma}{2\lambda}e_1$, and the cost of being attached to the whole network, $\frac{1}{2}(e_1 + e_3 + e_4)$.

- The Star:

$$\begin{aligned} e_1 &= \left[\left(\frac{1-\pi}{2\lambda} \right) - \left(\frac{\delta}{2\lambda}(3) \right) + \left(\frac{\gamma}{2\lambda}(e_2 + e_3 + e_4) \right) - \left(\frac{1}{2}(e_2 + e_3 + e_4) \right) \right] \\ e_2 &= \left[\left(\frac{1-\pi}{2\lambda} \right) - \left(\frac{\delta}{2\lambda}(1) \right) + \left(\frac{\gamma}{2\lambda}e_1 \right) - \left(\frac{1}{2}(e_1 + e_3 + e_4) \right) \right] \\ e_3 &= \left[\left(\frac{1-\pi}{2\lambda} \right) - \left(\frac{\delta}{2\lambda}(1) \right) + \left(\frac{\gamma}{2\lambda}e_1 \right) - \left(\frac{1}{2}(e_1 + e_2 + e_4) \right) \right] \\ e_4 &= \left[\left(\frac{1-\pi}{2\lambda} \right) - \left(\frac{\delta}{2\lambda}(1) \right) + \left(\frac{\gamma}{2\lambda}e_1 \right) - \left(\frac{1}{2}(e_1 + e_2 + e_3) \right) \right] \end{aligned}$$

The solution for the equilibrium values of the e_i are displayed below. There is one solution for e_1 , and, of course, the same symmetric solutions for e_2, e_3 and e_4 . The solution is:

$$\left[\begin{array}{l} e_1 = -\frac{1}{5\lambda^2 - 3\gamma^2 + 6\lambda\gamma} (-\lambda - 3\gamma + \pi\lambda + 3\pi\gamma + 9\lambda\delta + 3\gamma\delta), \\ e_2 = \frac{1}{5\lambda^2 - 3\gamma^2 + 6\lambda\gamma} (\lambda + \gamma - \pi\lambda - \pi\gamma + \lambda\delta - 3\gamma\delta), \\ e_3 = \frac{1}{5\lambda^2 - 3\gamma^2 + 6\lambda\gamma} (\lambda + \gamma - \pi\lambda - \pi\gamma + \lambda\delta - 3\gamma\delta), \\ e_4 = \frac{1}{5\lambda^2 - 3\gamma^2 + 6\lambda\gamma} (\lambda + \gamma - \pi\lambda - \pi\gamma + \lambda\delta - 3\gamma\delta) \end{array} \right]$$

if $2\lambda^2 - \gamma^2 + 2\lambda\gamma \neq 0$ and $3\lambda^2 - \gamma^2 + 2\lambda\gamma \neq 0$ and $5\lambda^2 - 3\gamma^2 + 6\lambda\gamma \neq 0$. Similarly, we obtain the solutions for other network types.

- The Wheel – the solution is:

$$\left[\begin{array}{l} e_1 = \frac{1}{-5\lambda + 2\gamma} (\pi + 2\delta - 1), \\ e_2 = \frac{1}{-5\lambda + 2\gamma} (\pi + 2\delta - 1), \\ e_3 = \frac{1}{-5\lambda + 2\gamma} (\pi + 2\delta - 1), \\ e_4 = \frac{1}{-5\lambda + 2\gamma} (\pi + 2\delta - 1) \end{array} \right]$$

if $2\lambda^2 - \gamma^2 + 2\lambda\gamma \neq 0$ and $3\lambda^2 - \gamma^2 + 2\lambda\gamma \neq 0$ and $5\lambda^2 - 4\gamma^2 + 8\lambda\gamma \neq 0$.

- The Full network – the solution is:

$$\begin{bmatrix} e_1 = \frac{1}{-5\lambda+3\gamma} (\pi + 3\delta - 1), \\ e_2 = \frac{1}{-5\lambda+3\gamma} (\pi + 3\delta - 1), \\ e_3 = \frac{1}{-5\lambda+3\gamma} (\pi + 3\delta - 1), \\ e_4 = \frac{1}{-5\lambda+3\gamma} (\pi + 3\delta - 1) \end{bmatrix}$$

if $3\lambda^2 - \gamma^2 + 2\lambda\gamma \neq 0$ and $5\lambda^2 - 3\gamma^2 + 2\lambda\gamma \neq 0$ and $-2\lambda^2 + \gamma^2 - \lambda\gamma \neq 0$

- The Kite/Cell – the solution is:

$$\begin{bmatrix} e_1 = -\frac{1}{5\lambda^3+\gamma^3-5\lambda\gamma^2+3\lambda^2\gamma} (-\lambda^2 - \lambda\gamma + \pi\lambda^2 + 2\lambda^2\delta - \gamma^2\delta + \pi\lambda\gamma + 5\lambda\gamma\delta), \\ e_2 = \frac{1}{5\lambda^3+\gamma^3-5\lambda\gamma^2+3\lambda^2\gamma} (\lambda^2 - \gamma^2 + 2\lambda\gamma - \pi\lambda^2 + \pi\gamma^2 - 7\lambda^2\delta + \gamma^2\delta - 2\pi\lambda\gamma), \\ e_3 = -\frac{1}{5\lambda^3+\gamma^3-5\lambda\gamma^2+3\lambda^2\gamma} (-\lambda^2 - \lambda\gamma + \pi\lambda^2 + 2\lambda^2\delta - \gamma^2\delta + \pi\lambda\gamma + 5\lambda\gamma\delta), \\ e_4 = \frac{1}{5\lambda^3+\gamma^3-5\lambda\gamma^2+3\lambda^2\gamma} (\lambda^2 - \gamma^2 - \pi\lambda^2 + \pi\gamma^2 + 3\lambda^2\delta + \gamma^2\delta) \end{bmatrix}$$

if $3\lambda^2 - \gamma^2 + 2\lambda\gamma \neq 0$ and $5\lambda^3 + \gamma^3 - 5\lambda\gamma^2 + 3\lambda^2\gamma \neq 0$ and $-2\lambda^2 + \gamma^2 - \lambda\gamma \neq 0$.

- The Line – the solution is:

$$\begin{bmatrix} e_1 = \frac{1}{5\lambda^2-\gamma^2+\lambda\gamma} (\lambda - \pi\lambda + \lambda\delta - \gamma\delta), \\ e_2 = -\frac{1}{5\lambda^2-\gamma^2+\lambda\gamma} (-\lambda - \gamma + \pi\lambda + \pi\gamma + 4\lambda\delta + \gamma\delta), \\ e_3 = -\frac{1}{5\lambda^2-\gamma^2+\lambda\gamma} (-\lambda - \gamma + \pi\lambda + \pi\gamma + 4\lambda\delta + \gamma\delta), \\ e_4 = \frac{1}{5\lambda^2-\gamma^2+\lambda\gamma} (\lambda - \pi\lambda + \lambda\delta - \gamma\delta) \end{bmatrix}$$

if $2\lambda^2 - \gamma^2 + 2\lambda\gamma \neq 0$ and $3\lambda^2 - \gamma^2 + 2\lambda\gamma \neq 0$ and $5\lambda^4 + \gamma^4 - 5\lambda^2\gamma^2 - 2\lambda\gamma^3 + 6\lambda^3\gamma \neq 0$.