Surplus Efficiency of Ex Ante Investments in Matching Markets with Nontransferabilities*

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Abstract

Does a competitive equilibrium in a matching market provide adequate incentives for investments made before the market when utility is not perfectly transferable? This paper derives a necessary and sufficient condition for equilibrium investments to maximize surplus conditional on the matching assignment in a one-sided market. Surplus efficiency of equilibrium payoffs ex post alone is sufficient for surplus efficient investments only when the equal treatment property holds in equilibrium. Sufficient (but not full) utility transferability in a well defined sense ensures this will hold and that a social planner who can only change investments cannot achieve higher aggregate surplus than the market.

Keywords: Matching, assignment models, investments, nontransferable utility, graph theory.

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1 Introduction

Do equilibrium allocations and payoffs in matching markets provide adequate incentives for investments in attributes that are relevant to matching partners and are made before the market? This is a relevant question, in

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particular for policy discussion concerning education acquisition and labor markets. For instance, there appear to be widespread concerns that – possibly because of asymmetric information (see Bénabou and Tirole, 2012) – salaries in the labor market may not adequately reflect the social marginal benefit of employees' productivity or human capital, distorting incentives for education investment. Similarly, when admission to good schools and colleges, and thus access to high quality peers in the classroom, is based partly on parents' income through user fees or house prices in the presence of borrowing constraints, rewards to prior effort in education acquisition or early childhood investments will be distorted.

The question has attracted considerable attention in the literature. Cole et al. (2001b) show that surplus efficient investments are in the equilibrium set when utility is perfectly transferable.[1] At the other extreme, for strictly nontransferable utility, such that surplus has to be split equally among partners, Peters and Siow (2002) establish Pareto (though not necessarily surplus) efficiency of investments in a two-sided matching market, though Bhaskar and Hopkins (2011) points out the limits of this results. On the other hand, Gall et al. (2006) provide an example of surplus inefficient investments in a one-sided market when utility is less than perfectly transferable and distorts the matching pattern. Gall et al. (2009) finds investment distortions generating surplus inefficiency in form of simultaneous over-investment at the top and under-investment at the bottom, and analyze rematching policies. Mailath et al. (2012) examine the relation of the dimensionality of the price system and potential investment distortions in a two-sided market. This raises the question of the degree of utility transferability required to ensure that investments maximize aggregate surplus. Evaluating allocations in terms of surplus efficiency appears reasonable from a normative, ex ante perspective (in the sense of Harsanyi, 1953), and from a positive point of view when surplus relates to output.

In essence, nontransferable utility may distort ex ante investments away from the surplus maximizing allocation through three possible channels. First, with non-transferabilities equilibrium payoffs may not maximize joint

[1] This is approximately true in finite economies (Cole et al., 2001a, Felli and Roberts, 2002).

surplus in each match formed in equilibrium. That is, there is ex post inefficiency in that given an equilibrium assignment and investments, in some matches a different division of the surplus may increase joint surplus. Second, payoff distortions may cause the equilibrium assignment to differ from the first best. That is, given equilibrium investments and payoffs there may be another assignment that is not stable but generates higher aggregate surplus, which may affect incentives. Third, given an equilibrium assignment equilibrium payoffs may not adequately reflect the externalities that an agent's investment generates on potential matches. That is, given the equilibrium assignment and payoffs, changing individual investments may generate higher aggregate surplus.

This paper shall be concerned with the first and the third channel only; analyzing surplus efficiency of the matching pattern and possible remedies is done elsewhere (Gall et al., 2009). First, a necessary and sufficient condition is derived that ensures equilibrium investments are surplus efficient conditional on the matching assignment. This is quite relevant, for instance, if policy determines the matching pattern. Essentially the condition requires matching market equilibrium payoffs to coincide with equilibrium payoffs when utility is fully transferable, which is stronger than requiring that payoffs maximize joint surplus in each match formed, i.e., ex post surplus efficiency.

Surplus efficiency requires payoffs to reflect the social benefit of investments in a first best world. A less demanding criterion is whether a social planner who can change investments, but otherwise remains constrained by nontransferabilities, can increase aggregate surplus. Indeed, equilibrium market prices for attributes correctly reflect the actual externalities (subject to nontransferabilities) of a rematch triggered by a change in investments if, and only if, equal treatment holds, that is, each attribute obtains the same payoff in every equilibrium match, independent of the attribute it is assigned to. With equal treatment the law of one price holds, and the market payoff for any attribute in any match reflects the actual opportunity cost of foregoing a different match for that attribute. Otherwise, some individuals will receive more than their true opportunity cost, which distorts investments. Hence, given a matching pattern, a social planner who can affect only individual investments cannot increase aggregate surplus beyond what is achieved by equilibrium investments induced by a matching market equilibrium that satisfies equal treatment. This finding extends to cases where the matching pattern reacts to changes in investments.

The condition can be readily applied. For instance, one-sided markets with strictly nontransferable utility (when surplus in a match has to be shared according to a certain, match-specific ratio, e.g. due to ex post renegotiations) often have full segregation in equilibrium (i.e. only matches of agents with the same attributes occur), which trivially implies the equal treatment property. Hence, investments in such markets are typically constrained efficient, conditional on equilibrium assignment and payoffs.[2]

The equal treatment property can be tied to transferability: a sufficient condition for equal treatment is that utility is transferable enough to allow partners in a match to transfer utility to another at a bounded, strictly positive rate in each possible match of attributes. Strictly nontransferable utility that yields some heterogeneity in the equilibrium matching typically causes equal treatment to fail, however. In this case ex post surplus efficiency of equilibrium payoffs does not imply surplus efficiency of investments. Intuitively, when equilibrium payoffs for an attribute differ for different attribute matches, the externalities generated by an agent's change in investment is not correctly reflected in payoffs as the law of one price fails.

The results are derived in a model of ex ante investments, made before a one-sided matching market with a continuum of agents. Costly investment determines the probability distribution over possible attributes an agent may attain. After attributes have realized agents enter the market, match into pairs, and jointly generate surplus, which depends on attributes. A matching equilibrium is a stable match with side payments, and equilibrium investments are optimal anticipating the matching equilibrium payoffs. Side payments are subject to nontransferabilities, captured by the Pareto frontier in each match, which may take any form between fully transferable and strictly nontransferable utility. Surplus in a match may not be monotone and transferability may vary between different attribute pairs, allowing for substantial heterogeneity in preference over possible matches and preferences for

[2] This extends to two-sided models when both market sides have the same type distribution as in the example of Peters and Siow (2002).

attributes need not be aligned.[3] The results follow from deriving the graph structure of the payoff externalities of a change in investments and using the structural properties of an equilibrium match of attributes.

The paper is organized as follows. Section 2 lays out the model and contains a preliminary result when there is full segregation in equilibrium. The general case is treated in Section 3 deriving the relation of equal treatment and constrained surplus efficient investments. Section 4 concludes and the appendix contains proofs and details omitted in the text.

2 A model of matching and investments

An economy is populated by a continuum of agents I of measure one. Agents are characterized by a type $\theta \in \Theta$ where Θ denotes a finite set of types. Before the match agents spend effort $e_i \in [\underline{e}, \bar{e}]$ with $0 \leq \underline{e} < \bar{e}$ at cost $c(e_i, \theta)$. The cost function is strictly increasing in θ and e, strictly convex and differentiable in e and satisfies $c(\underline{e}, \theta) = 0$ for all $\theta \in \Theta$. An individual's attribute $a \in A$, where A denotes a finite set, is stochastic and depends on effort: exerting effort e_i yields probability $p(a, e_i)$ of attaining an attribute a.[4] Attribute draws are independent across individuals.

Assumption 1 (Investment Technology)**.** *Suppose that*

(i) for all $e_i \in [\underline{e}, \bar{e}]$, $\sum_{a \in A} p(a, e_i) = 1$ and $p(a, e_i) > 0$ for all $a \in A$ (probability distribution with full support),

(ii) $p(a, e_i)$ is strictly monotone, concave, and differentiable in e_i for all $a \in A$,

For instance, if $A = \{a_0, a_1\}$ an investment technology that satisfies this assumption is $p(a_0, e_i) = e_i$ with $e_i \in [\epsilon, 1 - \epsilon]$ for $\epsilon \in (0, 1/2)$.[5]

[3] See also Dizdar (2012) for a recent extension of the efficiency result by Cole et al. (2001b) to multidimensional types and allowing for payoffs that are not supermodular.

[4] This reduces the problem of multiple equilibria due to coordination failure as noted by Bhaskar and Hopkins (2011) and has been used e.g. in Gall et al. (2009).

[5] The form of $p(a, e_i)$ is chosen for simplicity. Using a technology that allows to choose a portfolio of efforts, e.g. one for each attribute with a resource constraint, appears not to affect the derivation of the results below.

Invoking a law of large numbers, denote the realized measure of an attribute $a \in A$ in the matching market given investments $e = (e_i)_{i \in I}$ by

$$q(a,e) = \int_{i \in I} p(a, e_i) di.$$

Once attributes have realized agents match into pairs. Unmatched agents obtain payoff 0, and a matched pair of agents (i, j) jointly generates surplus of at most $y(a_i, a_j)$. Note that no order is imposed on A, that is, $y(a, a')$ may not be monotone in its arguments, thus potentially allowing for multi-dimensional attributes. In most relevant applications expected surplus will monotonically increase as own effort investment increases, assume therefore

$$\sum_{a \in A} y(a, a') \frac{\partial p(a, e_i)}{\partial e_i} > 0 \text{ for all } a' \in A. \tag{1}$$

Aggregate surplus in a match (i, j) may depend on its division among partners (for instance due to moral hazard problems in the match, limited liability, or behavioral concerns), so that individual payoffs u_i and u_j satisfy

$$u_i \leq \phi(a_i, a_j, u_j) \text{ with } u_i + \phi(a_i, a_j, u_j) \leq y(a_i, a_j).$$

$\phi(a_i, a_j, u_j)$ is the Pareto or utility possibility frontier in a match (i, j), giving i's maximum payoff when j receives u_j given attributes a_i and a_j (the notation follows Legros and Newman, 2007). Suppose $\phi(a_i, a_j, u_j)$ is continuous and weakly decreases in u_j with $\phi(a_i, a_j, 0) > 0$ and $\phi(a_i, a_j, u) = 0$ implies $u > 0$ for all $a_i, a_j \in A$. Since transferability may depend on the match of attributes (a, a') some combinations may allow for full transferability, while others do not. This may be a source of gains from trades, as more transferability with some partner than with another may compensate for lower maximal joint surplus. Since the market is one-sided, partners in a match may switch roles so that the Pareto frontier has to be symmetric, $\phi(a, a', u) = \phi(a', a, u)$.

To see that $\phi(.)$ captures the degree of payoff transferability in a match, note that full transferability of utility corresponds to

$$\phi(a, a', u) = y(a, a') - u, \text{ for } 0 \leq u \leq y(a, a').$$

At the other extreme is strictly nontransferable utility, e.g., if joint surplus has to be shared at a ratio $\delta \in [1/2, 1]$. Then $\phi(a, a', u_j) = \delta y(a, a')$ for $u_j \leq (1-\delta)y(a, a')$, and $\phi(a, a', u_j) = (1-\delta)y(a, a')$ for $(1-\delta)y(a, a') < u_j \leq \delta y(a, a')$. The ratio could, of course, depend on the match.

Equilibrium Concept

A *matching market equilibrium* $(u^*, \mathcal{P}^*)$ are payoffs $u^* = (u_i^*)_{i \in I}$ and a partition $\mathcal{P}^*$ of I into pairs preserving the measures of attributes $q(a, e)$, such that there are no agents $i, j \neq i \in I$ with $(i, j) \notin \mathcal{P}^*$ and payoffs u_i, u_j such that $u_i \leq \phi(a_i, a_j, u_j)$ and both $u_i > u_i^*$ and $u_j > u_j^*$. Measure consistency ensures the measure of first members of matched attribute pairs (a, a') equals the measure of the second one. An *investment cum matching equilibrium* are investments $e^* = (e_i^*)_{i \in I}$ and a matching market equilibrium $(u^*, \mathcal{P}^*)$ given the measures of attributes $q(a, e)$ induced by e, such that no individual $i \in I$ can obtain strictly higher expected payoff given the matching equilibrium $(u^*, \mathcal{P}^*)$ choosing investment $e_i' \neq e_i^*$.

Since matching is into pairs in a continuum economy, existence of a stable match is guaranteed (see for instance Kaneko and Wooders, 1986), determining individual payoffs given investments. This means that the investment stage is, in fact, an anonymous game, for which existence of an equilibrium has been established, for instance, by Mas-Colell (1984). Note that the investment cum matching equilibria relies on rational expectations of the matching equilibrium payoffs given aggregate investments. Therefore there may be multiple investment cum matching equilibria. Whether a matching equilibrium maximizes total surplus given the realized attributes depends on the properties of $y(a, a')$ and $\phi(a, a')$, see e.g. Legros and Newman (2007).

An equilibrium partition $\mathcal{P}^*$ characterizes an assignment of attributes μ that maps A into its power set, defined by $\mu(a) = \{a_j : a_i = a \wedge (i, j) \in \mathcal{P}^*\}$ for all $a \in A$. Denote the measure of an attribute pair (a, a') implied by μ by $\rho(a, a')$. Measures $\rho(a, a')$ are determined by the system of equations

$$q(a, e) = \sum_{a' \in \mu(a)} \rho(a, a') + 2\rho(a, a) \text{ for } a \in A. \tag{2}$$

Some form of rationing may be required when $\mu(a)$ has more than one element for some a. Assume that agents are assigned randomly with probabilities implied by the relative frequencies of matches (a, a') with $a' \in \mu(a)$: an agent with attribute a is assigned to an agent with attribute a' with probability $\rho(a, a')/q(a, e)$.

For example, let $A = \{a_1; a_2\}$ and suppose the matching equilibrium is not full segregation. If e.g. $q(a_1, e) > q(a_2, e)$, $\mu(a_1) = \{a_1; a_2\}$ and $\mu(a_2) = a_1$, measure of matches are $\rho(a_1, a_2) = q(a_2, e)$, $\rho(a_1, a_1) = q(a_1, e) - q(a_2, e)$, and $\rho(a_2, a_2) = 0$. Matching probabilities are then $\hat{\rho}(a_1, a_2) = q(a_2, e)/q(a_1, e)$ and $\hat{\rho}(a_1, a_1) = 1 - \rho(a_2, a_1)$, and $\hat{\rho}(a_2, a_1) = 1$ and $\hat{\rho}(a_2, a_2) = 0$.

Note that, when utility is not fully transferable, equal treatment may fail. A matching market equilibrium satisfies the equal treatment property, if each attribute obtains the same payoff no matter which other attribute it is matched to, i.e.

$$v(a) = \phi(a, a', v(a')) \text{ for all } a \neq a' \in A \text{ with } \rho(a, a') > 0. \tag{ET}$$

In case this fails even for homogenous matches (a, a), that is, two agents i and j with the same attribute obtain different equilibrium payoffs $u_i^* \neq u_j^*$ when matched together, assign equal probability to each possible equilibrium payoff division in a match.

Equilibrium Investments

An agent's equilibrium payoff given attribute a can be written as

$$v(a) = \sum_{a' \in A} \rho(a, a') E[\phi(a, a', u_{a'}^*)],$$

where $u_{a'}^*$ denotes equilibrium payoff u_j^* for an agent j with $a_j = a'$ in a match (a, a') and the expectation is with respect to $u_{a'}^*$ in case equal treatment fails for a homogenous match (a, a). Anticipating the market outcome agents choose effort investments. An agent i's choice of e_i therefore solves $\max_{e_i} \sum_{a \in A} p(a, e_i) v(a) - c(e_i, \theta_i)$. The equilibrium effort choice e_i^* satisfies

$$\sum_{a \in A} v(a) \frac{\partial p(a, e_i^*)}{\partial e_i} = \frac{\partial c(e_i^*, \theta_i)}{\partial e_i}. \tag{3}$$

E.g., if surplus has to be shared equally, $\rho(a, a) = 1$ and $u_a^* = \phi(a, a, u_a^*)$ implies that $v(a) = y(a, a)/2$.

Surplus Optimal Allocation

Compare this to effort investments that maximize aggregate surplus if a social planner can choose investment levels and the surplus sharing to ensure $u_i + u_j = y(a_i, a_j)$ in all matches (i, j) when the matching is given by $\mu(a)$.[6] The social planner solves

$$\max_{(e_i)_{i\in I}} \sum_{a\in A} \left(\rho(a,a)y(a,a) + \frac{1}{2} \sum_{a'\neq a\in A} \rho(a,a')y(a,a') \right) - \int_I c(e_i, \theta_i)di. \quad (4)$$

The optimization is over the investments e of a continuum of individuals, but note that the type space Θ is finite and the investment cost convex, while expected payoffs are the same for all individuals. Therefore all individuals i of the same type θ_i will necessarily have the same investment e_i in optimum, and the optimization is really only over a finite vector of investments.

Measures $\rho(a, a')$ depend on $q(a, e)$, and thus on e, through (2). If the equilibrium assignment μ remains constant, measures $\rho(a, a')$ are differentiable with respect to $q(a, e)$ and thus with respect to e_i. However, a marginal change in investment e may trigger a change in the equilibrium assignment μ, adding or subtracting a match (a, a') with surplus $y(a, a')$, thus altering (2) defining the measures $\rho(a, e)$. This will induce a discrete change in the marginal social benefit of investment. Focus for now on cases such that a marginal change of investments e does not affect μ. Call such equilibrium assignments static. Then investments that solve (4) must satisfy for each $i \in I$

$$\sum_{a\in A} \left(\frac{\partial \rho(a,a)}{\partial e_i} y(a,a) + \sum_{a'\neq a\in A} \frac{\partial \rho(a,a')}{\partial e_i} \frac{y(a,a')}{2} \right) = \frac{\partial c(e_i, \theta_i)}{\partial e_i} \quad (5)$$

Since the first derivative of expected surplus (1) decreases in investment e_i, as $p(a, e_i)$ is concave in e_i, (5) is sufficient as well, given the matching $\mu(a)$.

[6] If one is interested in surplus maximizing investments constrained on equilibrium payoffs, i.e. taking joint surplus in each match as given by the equilibrium payoffs, it suffices to substitute maximal surplus $y(a, a')$ with equilibrium surplus $\hat{y}(a_i, a_j) = u_i^* + u_j^*$ for all $a_i \in \mu(a_j)$ in the optimization problem.

Full Segregation

For instance, if surplus has to be split equally, matching takes the form of full segregation, $\mu(a) = a$ for all $a \in A$. If this remains the equilibrium assignment for any investments e, then investments are surplus efficient if, and only if, investments e_i^* given by (3) satisfy condition (5). Under full segregation $\rho(a,a) = q(a,e^*)/2$ and $\rho(a,a') = 0$ for all $a \neq a'$ and (5) becomes

$$\sum_{a \in A} \frac{y(a,a)}{2} \frac{\partial p(a,e_i)}{\partial e_i} = \frac{\partial c(e_i,\theta_i)}{\partial e_i}.$$

On the other hand, equal sharing of surplus implies that $\sum_{a \in A} (\frac{y(a,a)}{2} - v(a)) \frac{\partial p(a,e_i)}{\partial e_i} = 0$. Hence, if full segregation is an equilibrium and equal sharing of expected surplus maximizes joint surplus in each match (a,a), the social planner cannot increase aggregate surplus by choosing different investments or adjusting sharing rules. Therefore investments are surplus efficient. The following proposition sums up the argument.

Proposition 1 (Full Segregation)**.** *Given an assignment of attributes $\mu(a) = a$ and payoffs u^*, equilibrium attribute investments coincide with surplus maximizing investment levels if, and only if,*

$$\sum_{a \in A} \left(\frac{y(a,a)}{2} - E[\phi(a,a,u_a^*)] \right) \frac{\partial p(a,e_i)}{\partial e_i} = 0 \textit{ for all } i \in I,$$

This is implied by $E[\phi(a,a,u_a^*)] = y(a,a)/2$ for all $a \in A$.

The condition that equal division of the payoff is surplus efficient in a match of agents with equal attributes seems likely to be satisfied in many relevant applications. The following counterexample demonstrates that it may fail, however, although rather extreme assumptions are needed.

Example: moral hazard in partnerships

Assume that in a match (i,j) revenue $R(a_i,a_j)$ is realized with probability $g(x_i,x_j)$, depending on individual effort choices x_i and x_j as follows:

$$g(x_i,x_j) = x_i^{\alpha} x_j^{1-\alpha}.$$

Let $\alpha \geq 1/2$. Exerting effort an agent i incurs utility cost $x_i^2/2$. In a match agents may contract on the share s of the revenue that goes to i, but do not use monetary transfers, e.g. due to liquidity constraints. Hence,

$$u_i = s x_i^\alpha x_j^{1-\alpha} R(a_i, a_j) - x_i^2/2 \text{ and } u_j = (1-s) x_i^\alpha x_j^{1-\alpha} R(a_i, a_j) - x_j^2/2.$$

Individual optimal effort choice pins down effort levels depending on s, $x_i(s)$ and $x_j(s)$. Therefore individual payoffs depend also on s and are given by

$$u_i(s) = s(\alpha s)^\alpha ((1-\alpha)(1-s))^{1-\alpha} (1 - \alpha s/2) R(a_i, a_j)^2 \text{ and}$$
$$u_j(s) = (1-s)(\alpha s)^\alpha ((1-\alpha)(1-s))^{1-\alpha} (1 - (1-\alpha)(1-s)/2) R(a_i, a_j)^2.$$

That is, sharing rule s determines a pair of u_i and u_j and thus the joint surplus in match (i, j). This can be used to construct the Pareto frontier,

$$\phi(a_i, a_j, u) = \arg\max_s u_i(s) \text{ s.t. } u_j(s) \geq u.$$

Denote the sharing rule that maximizes joint surplus in a match (i, j) by $s^* = \arg\max_s u_i(s) + u_j(s)$, and the maximum joint surplus by $y(a_i, a_j) = u_i(s^*) + u_j(s^*)$.

On the other hand, denote the sharing rule that allows equal sharing of the joint surplus, such that that $u_i(\hat{s}) = u_j(\hat{s})$, by $\hat{s}$. Sharing the joint surplus equally also maximizes joint surplus if and only if the effort investment problem is symmetric, i.e. $\hat{s} = s^*$ if, and only if $\alpha = 1/2$.

Figure 1 depicts $\phi(a_i, a_j, u)$ for three different matches (a, a), (a, a'), and (a', a') with $a > a'$. The 45° line pins down payoffs for equal sharing and the dashed lines indicate the surplus maximizing payoff sharing.

Suppose that $\mu(a) = a$ in equilibrium (this is implied by e.g. $R(a, a) - R(a, a')$ sufficiently high for all $a > a'$, see Appendix). To check the condition in Proposition 1, note that, whenever $\alpha > 1/2$, for all $a \in A$

$$y(a, a)/2 - E[\phi(a, a, u_a^*)] = \kappa(\alpha) R(a, a)^2,$$

for a constant $\kappa(\alpha) > 0$ depending only on α. Therefore investments are not surplus efficient unless the investment technology $(\partial p(a, e_i)/\partial e_i)$ exactly compensates the differences $\kappa(\alpha) R(a, a)^2$.

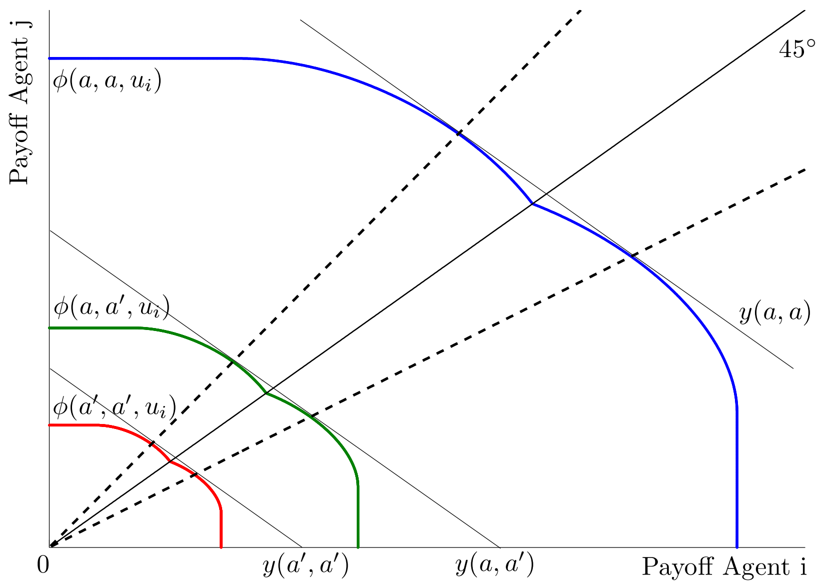

Figure 1: Utility Possibility Frontiers

Hence, a social planner who could enforce a different surplus distribution than equal sharing can increase aggregate surplus. Suppose that $s^p = \hat{s} + \epsilon$ is enforced for each match (i,j). If $R(a,a) - R(a,a')$ is sufficiently high for all $a > a'$ full segregation remains the equilibrium outcome and thus $u_i(s^p) + u_j(s^p) > u_i(\hat{s}) + u_j(\hat{s})$ for all matches (i,j). This decreases the difference $y(a,a)/2 - E[\phi(a,a,u_a^*)]$ and increases both aggregate surplus and investments.

3 Heterogeneous Matches

Allow now for equilibrium assignment of attributes μ that do not induce full segregation, i.e. $\mu(a) \neq a$ for some attribute a. Now changing investments also changes the expected equilibrium match of at least some attributes, affecting either only the measure of matched attribute pairs $\rho(a,a')$ while μ remains unchanged, or affecting both $\rho(.)$ and μ.

The set of attributes and measures $\rho(.)$ define an undirected, weighted graph G with a set of vertices A, a set of edges $E = \{(a,a') : a' \in \mu(a)\}$, and the weights of edges (a,a') given by $\rho(a,a')$. Let C denote the set of connected components in G. For instance, if surplus has to be shared equally (i.e. $\delta = 1/2$), this implies $\mu(a) = a$ (full segregation) and equilibrium payoffs $u_i^* = y(a_i, a_i)/2$. Therefore each vertex $a \in A$ has only one edge, (a, a), and is a connected component, so that the set of components is $C = A$.

Here we limit our attention to graphs whose connected components $c \in C$ have at most as many edges as vertices. Uniqueness of the matching equilibrium implies this property. Otherwise, the surplus maximizing matching market equilibrium given attributes necessarily has this property. See appendix for details on this and the next statement. Denote the sets of vertices and edges in c by A^c and E^c. This then implies the following fact.

Fact 1. *In a graph G associated to an equilibrium assignment μ such that $|E^c| \leq |A^c|$ for each connected component c, in each connected component either*

(i) there is exactly one vertex $a \in A^c$ with $a \in \mu(a)$ and c does not contain a cycle, or

(ii) $a \notin \mu(a)$ for all $a \in A^c$ and $|A^c| = |E^c|$, then c contains one cycle of n vertices $\{a_1; ...; A_n\}$ and edges (a_n, a_1) and (a_i, a_{i+1}) for $i = 1, .., n-1$, or

(iii) $a \notin \mu(a)$ for all $a \in A^c$ and $|A^c| > |E^c|$, then c has vertices $\{a_1; ...; A_n\}$ and edges and (a_i, a_{i+1}) for $i = 1, .., n-1$.

That is, for each component either $|A^c| = |E^c|$ and c contains a cycle or an edge (a, a) (which is a cycle of length 0), or $|A^c| > |E^c|$ and c has at least two terminal vertices.

This observation can be tied to whether or not the equilibrium assignment μ will respond to a change in investments e. Recall that edge weights $\rho(.)$ are defined by the system of equations (2). $|A| > |E|$ is equivalent to G having a component with vertices $\{a_1; ...; A_n\}$ and edges and (a_i, a_{i+1}) for $i = 1, .., n-1$. Then for this component the weights solve

$$\begin{aligned}
&\rho(a_i, a_{i-1}) = q(a_i) - \rho(a_{i+1}, a_i) \text{ for } i = 2, ..., n-1, \text{ and} \\
&\rho(a_1, a_2) = q(a_1) \text{ and } \rho(a_{n-1}, a_n) = q(a_n).
\end{aligned}$$

This means

$$\sum_{i=0}^{n}(-1)^{i+1}q(a_i,e)=0.$$

Therefore a marginal change of investments e, changing measures $q(a,e)$ in turn, will typically violate the condition and the equilibrium assignment μ must change (adding new or removing old matches at a_0 or a_n) in response to a change in investments. This yields the following statement.

Fact 2. *If, and only if, in the graph G associated to an equilibrium assignment μ the number of vertices strictly exceeds the number of edges, $|A|>|E|$, then a marginal change in investment e implies that μ is no longer an equilibrium assignment.*

The following definition characterizes equilibrium assignments μ that do not change in response to a marginal change in investments.

Definition 1. *An equilibrium assignment μ is static if the number of edges in the graph G induced by μ at least equals the number of vertices.*

Intuitively, whenever component c contains as many edges as vertices, it contains a cycle or a vertex $a\in A^c$ with $a\in\mu(a)$, so that any marginal change in measures $q(a,e)$ can be accommodated by adjusting weights of edges in the cycle, or $\rho(a,a)$, without needing to adjust the graph. In particular, the assignment $\mu(a)=a$ for all $a\in A$, i.e. full segregation as above, is static.

For instance, suppose that $A=\{a_1;a_2\}$ and the equilibrium assignment is $\mu(a_1)=\{a_1;a_2\}$ and $\mu(a_2)=a_1$, inducing $\rho(a_1,a_2)=q(a_2)/q(a_1)$. Then graph G contains one connected component, G, which in turn contains two edges and two vertices. A marginal change in e marginally changes $q(a_1)$ and $q(a_2)$, and therefore $\rho(a_1,a_2)$, but any change in $\rho(a_1,a_2)$ is counterbalanced by $\rho(a_1,a_1)=1-\rho(a_1,a_2)$. If the equilibrium assignment is $\mu(a_1)=a_2$ and $\mu(a_2)=a_1$, however, the single connected component of G contains only one edge (a_1,a_2), but two vertices. Then a marginal change in e changes $q(a_1)$ and $q(a_2)$ and, since $\sum\frac{\partial p(a_i,e_i)}{\partial e_i}=0$, the matching given by μ, with $\rho(a_1,a_2)=1/2$ relying on $q(a_1,e)=q(a_2,e)$, becomes impossible.

3.1 Static Assignments

Focus for now on static equilibrium assignments μ. In this case (5) is a sufficient and necessary condition for a solution of the social planner's problem (4). The LHS of (5) can be decomposed into the set of disjoint connected components C, since by definition $\rho(a, a') = 0$ for any $a \in A^c$ and $a' \notin A^c$:

$$\sum_{c \in C} \sum_{a \in A^c} \left(\frac{\partial \rho(a,a)}{\partial e_i} y(a,a) + \sum_{a' \neq a \in A^c} \frac{\partial \rho(a,a')}{\partial e_i} \frac{y(a,a')}{2} \right).$$

The following analysis will distinguish between cases (i) and (ii) of Fact 1.

Case 1: no cycles

Start with case (i), i.e. focus on a component c that contains exactly one $a \in A^c$ with $a \in \mu(a)$; denote it by a_0. Define the distance $d(a, a')$ of two vertices $a, a' \in c$ by the number of edges in the shortest path connecting them, e.g. $d(a, a') = 1$ if and only if $a' \in \mu(a)$. Let $n = \max_{a \in A^c} d(a_0, a)$ the maximum distance from vertex a_0. Define by $A_i^c = \{a \in A^c : d(a_0, a) = i\}$ the set of vertices that have common distance i from a_0. Figure 2 shows an example.

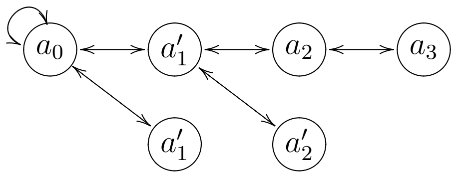

Figure 2: Example for a component c with one vertex a_0 that links to itself.

The effects of a change of investments on attributes in component c,

$$\sigma^c = \sum_{a \in A^c} \left(\frac{\partial \rho(a,a)}{\partial e_i} y(a,a) + \sum_{a' \neq a \in A^c} \frac{\partial \rho(a,a')}{\partial e_i} \frac{y(a,a')}{2} \right)$$

can be derived by summing up the effects on each match (a, a') in c ordered by their distance from a_0:

$$\sigma^c = y(a_0,a_0) \frac{\partial \rho(a_0,a_0)}{\partial e_i} + \sum_{i=1}^{n} \left(\sum_{a_i \in A_i^c} \sum_{a_{i-1} \in A_{i-1}^c \cap \mu(a_i)} y(a_{i-1},a_i) \frac{\partial \rho(a_{i-1},a_i)}{\partial e_i} \right).$$

Let $a_i \in A_i^c$ and $a_{i-1} \in A_{i-1}^c$. Then $\rho(a_{i-1}, a_i) = q(a_i, e)$ if a_i is a terminal vertex, i.e. $\mu(a_i) = a_{i-1}$, and for a_i, $i \geq 1$, that are not terminal vertices

$$\rho(a_{i-1}, a_i) = q(a_i, e) - \sum_{a_{i+1} \in \mu(a_i) \cap A_{i+1}^c} \rho(a_i, a_{i+1}).$$

Finally, $\rho(a_0, a_0) = \mu(a_0)/2 - \sum_{a_1 \in A_1^c} \rho(a_0, a_1)$. These observations imply

$$\begin{aligned}\sigma^c =& \frac{y(a_0, a_0)}{2} \frac{\partial p(a_0, e_i)}{\partial e_i} + \sum_{a_1 \in A_1^c} \left(y(a_0, a_1) - \frac{y(a_0, a_0)}{2} \right) \frac{\partial p(a_1, e_i)}{\partial e_i} \\ &+ \sum_{a_2 \in A_2^c} \sum_{a_1 \in A_1^c \cap \mu(a_2)} \left(y(a_1, a_2) - y(a_0, a_1) + \frac{y(a_0, a_0)}{2} \right) \frac{\partial p(a_2, e_i)}{\partial e_i} + ... + \\ &+ \sum_{a_n \in A_n^c} \sum_{a_{n-1} \in A_{n-1} \cap \mu(a_n)} \left(y(a_{n-1}, a_n) - ...(-1)^n \frac{y(a_0, a_0)}{2} \right) \frac{\partial p(a_n, e_i)}{\partial e_i}.\end{aligned}$$

Define the "externality" that vertices closer to a_0 have on those further apart by

$$x(a_i) = y(a_{i-1}, a_i) - x(a_{i-1}) \text{ for } i = 1, ..., n, \tag{6}$$

and $x(a_0) = y(a_0, a_0)/2$. Then

$$\sigma^c = \frac{y(a_0, a_0)}{2} \frac{\partial p(a_0, e_i)}{\partial e_i} + \sum_{i=1}^{n} \left(\sum_{a_i \in A_i^c} \sum_{a_{i-1} \in A_{i-1}^c \cap \mu(a_i)} (y(a_{i-1}, a_i) - x(a_{i-1})) \frac{\partial p(a_i, e_i)}{\partial e_i} \right). \tag{7}$$

To verify whether surplus efficient investments coincide with equilibrium investments recall that the latter were determined by

$$\sum_{c \in C} \sum_{a \in A^c} v(a) \frac{\partial p(a, e_i^*)}{\partial e_i} = \frac{\partial c(a, e_i^*)}{\partial e_i}.$$

Hence, for each component c

$$\sigma^c = \sum_{a \in A^c} v(a) \frac{\partial p(a, e_i^*)}{\partial e_i}$$

is equivalent to

$$\begin{aligned}&\sum_{i=1}^{n} \left(\sum_{a_i \in A_i^c} \sum_{a_{i-1} \in A_{i-1}^c \cap \mu(a_i)} (y(a_{i-1}, a_i) - x(a_{i-1}) - v(a_i)) \frac{\partial p(a_i, e_i^*)}{\partial e_i} \right) \\ &+ \left(\frac{y(a_0, a_0)}{2} - v(a_0) \right) \frac{\partial p(a_0, e_i^*)}{\partial e_i} = 0.\end{aligned} \tag{8}$$

This is implied by $v(a_i) = y(a_{i-1}, a_i) - x(a_{i-1})$ for all $a_i \in A_i^C$ and $a_{i-1} \in A_{i-1}^c$ for all distances $i = 1, ..., n$. Since by definition $x(a_i) = y(a_{i-1}, a_i) - x(a_{i-1})$, this means that

$$v(a_i) = y(a_{i-1}, a_i) - v(a_{i-1}) \text{ for } i > 0, \text{ and } v(a_0) = y(a_0, a_0)/2, \qquad (9)$$

implies (8). Note that (9) characterizes the equilibrium payoffs supporting a stable matching under fully transferable utility. Hence, if equilibrium payoffs do not coincide with the payoffs in a matching market equilibrium under fully transferable utility, the condition $v(a_i) = y(a_{i-1}, a_i) - x(a_{i-1})$ will fail for some attributes. Unless distortions for some attribute a_j with $j > i$ exactly compensate this, (8) will fail. Note that even if for some attributes a_i and a_j the respective distortions in payoff exactly offset each other, this will no longer be the case for a slight change of $\frac{\partial p(a, e_i)}{\partial e_i}$, that is, a marginal perturbation of the investment technology.

Case 2: cycles

The case when c has a cycle is a generalization of the one above, by allowing for cycles that have length greater than 0. Define the distance $d^c(a)$ of a vertex $a \in A^c$ to the cycle by the number of edges in the shortest path connecting them, e.g. $d^c(a) = 0$ if and only if a is part of the cycle. Let $n = \max_{a \in A^c} d^c(a)$ the maximum distance from the cycle. Define by $A_i^c = \{a \in A^c : d^c(a) = i\}$ the set of vertices that have common distance i from the cycle. Figure 3 shows an example.

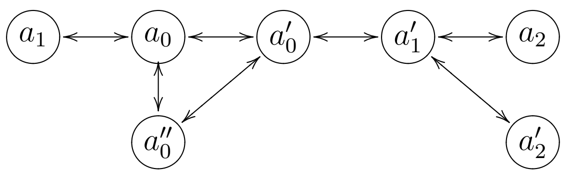

Figure 3: Example for a component c with a cycle (vertices a_0, a_0', and a_0'').

Again the effects of a change of investments on attributes in component c, σ^c can be derived by summing up the effects on each match (a, a') in c

ordered by their distance from the cycle:

$$\sigma^c = \sum_{a_0 \in A_0^c} \sum_{a_0' \in \mu(a_0) \cap A_0^c} \frac{y(a_0, a_0')}{2} \frac{\partial \rho(a_0, a_0')}{\partial e_i}$$
$$+ \sum_{i=1}^{n} \left(\sum_{a_i \in A_i^c} \sum_{a_{i-1} \in A_{i-1}^c \cap \mu(a_i)} y(a_{i-1}, a_i) \frac{\partial \rho(a_{i-1}, a_i)}{\partial e_i} \right).$$

Let $a_i \in A_i^c$ and $a_{i-1} \in A_{i-1}^c$. Then $\rho(a_{i-1}, a_i) = q(a_i, e)$ if a_i is a terminal vertex, i.e. $\mu(a_i) = a_{i-1}$. $\rho(a_{i-1}, a_i) = q(a_i, e) - \sum_{a_{i+1} \in \mu(a_i) \cap A_{i+1}^c} \rho(a_i, a_{i+1})$ and for vertices a_0 and $a_0', a_0'' \in \mu(a_0)$ in the cycle, it must hold that $\rho(a_0, a_0') + \rho(a_0, a_0'') = \mu(a_0) - \sum_{a_1 \in \mu(a_0) \cap A_1^c}$.

Denote by $n^0 = (|A_0^c| + 1)/2$ the maximum distance between any two vertices on the cycle. Then the "externality" that vertices closer to a_0 have on those further apart can be expressed as

$$x(a_0) = \frac{1}{2} \sum_{i=0}^{n^0 - 1} (-1)^i \sum_{a, a' \in A_0^c :\, d(a, a_0) = d(a', a_0) - 1} y(a, a'), \tag{10}$$

and using the definition of $x(a_i)$ above,

$$\sigma^c = \sum_{a_0 \in A_0^c} x(a_0) \frac{\partial p(a_0, e_i)}{\partial e_i} + \sum_{i=1}^{n} \left(\sum_{a_i \in A_i^c} \sum_{a_{i-1} \in A_{i-1}^c \cap \mu(a_i)} (y(a_{i-1}, a_i) - x(a_{i-1})) \frac{\partial p(a_i, e_i)}{\partial e_i} \right). \tag{11}$$

This expression coincides with (7) if $A_0^c = a_0$, i.e. a cycle of length 0.

To verify whether equilibrium investments satisfy (5) note again that for each component c

$$\sigma^c = \sum_{a \in A^c} v(a) \frac{\partial p(a, e_i^*)}{\partial e_i}$$

is equivalent to

$$\sum_{i=1}^{n} \left(\sum_{a_i \in A_i^c} \sum_{a_{i-1} \in A_{i-1}^c \cap \mu(a_i)} (y(a_{i-1}, a_i) - x(a_{i-1}) - v(a_i)) \frac{\partial p(a_i, e_i^*)}{\partial e_i} \right)$$
$$+ \sum_{a_0 \in A_0^c} (x(a_0) - v(a_0)) \frac{\partial p(a_0, e_i^*)}{\partial e_i} = 0. \tag{12}$$

Note that (12) becomes (8) if the cycle only has one edge (a_0, a_0). Here $v(a_0) = x(a_0)$ holds if payoffs $v(a_0)$ solve the system of equations

$$v(a_0) = y(a_0, a_0') - v(a_0') \text{ for all } a_0, a_0' \in A_0^c \text{ with } a_0 \in \mu(a_0').$$

Repeating the argument made above, if equilibrium payoffs do not coincide with the payoffs in a matching market equilibrium under fully transferable utility, the conditions $v(a_i) = y(a_{i-1}, a_i) - x(a_{i-1})$ or $v(a_0) = x(a_0)$ will fail for some attributes. These arguments are summarized in the following proposition.

Proposition 2 (Static Assignments). *Suppose an equilibrium assignment μ is static. Then equilibrium investments coincide with the ones chosen by a surplus maximizing social planner if, and only if,*

$$\sum_{c\in C}\sum_{i=1}^{n}\left(\sum_{a_i\in A_i^c}\sum_{a_{i-1}\in A_{i-1}^c\cap\mu(a_i)}\Big(x(a_i) - \phi(a_{i-1}, a_i, u^*_{a_{i-1}})\Big)\frac{\partial p(a_i, e_i^*)}{\partial e_i}\right)$$
$$+\sum_{c\in C}\sum_{a_0\in A_0^c}(x(a_0) - v(a_0))\frac{\partial p(a_0, e_i^*)}{\partial e_i} = 0,$$

where $x(a_i)$ is defined by (6) and (10).

This condition is satisfied if equilibrium payoffs u^* coincide with equilibrium payoffs under fully transferable utility ($\phi(a, a', u) = y(a, a') - u$).

That is, if equilibrium payoffs coincide with those under perfectly transferable utility, then the equilibrium allocation coincides with the one chosen by the social planner. Otherwise, distortions in incentives that arise for some equilibrium matches of attributes typically matter in aggregate. In case payoff distortion for different attributes happen to exactly compensate each other, surplus efficiency is not robust to a marginal perturbation in the investment technology $p(a, e_i)$. Note that Proposition 2 implies Proposition 1. The following examples illustrates Proposition 2 and emphasizes the condition may fail despite surplus efficiency of equilibrium payoffs.

Example: Heterogenous Matches

Let $A = \{a_0; a_1\}$ and assume that surplus has to be shared according to sharing rule $u_a = \delta(a, a')y(a, a')$ and $\delta(a, a') = 1 - \delta(a, a')$. Let $\delta(a, a) = 1/2$

for all $a \in A$ for the sake of simplicity. Suppose that $\mu(a_0) = \{a_0; a_1\}$ and $\mu(a_1) = a_0$, to keep the notation similar to the one used above (the reverse case is analogous). This is consistent with a matching market equilibrium if

$$\frac{y(a_1, a_1)}{2y(a_0, a_1)} \leq \delta(a_1, a_0) \leq 1 - \frac{y(a_0, a_0)}{2y(a_0, a_1)}. \qquad \text{(GDD)}$$

The condition in Proposition 2 becomes

$$\left(y(a_0, a_1) - \frac{y(a_0, a_0)}{2} - \delta(a_1, a_0) y(a_0, a_1) \right) \frac{\partial p(a_1, e_i)}{\partial e_i} + \left(\frac{y(a_0, a_0)}{2} - \frac{y(a_0, a_0)}{2} \right) \frac{\partial p(a_0, e_i)}{\partial e_i} = 0.$$

Note that here $y(a_0, a_1) - y(a_0, a_0)$ gives the social marginal benefit of turning an a_0 attribute into an a_1 attribute (since $p(a_0, e_i) = 1 - p(a_1, e_i)$), i.e., exchanging an (a_0, a_0) match for an (a_0, a_1) match. Since utility is perfectly transferable in (a, a) matches the condition reduces to

$$[1 - \delta(a_1, a_0)] y(a_0, a_1) \frac{\partial p(a_1, e_i)}{\partial e_i} = \frac{y(a_0, a_0)}{2} \frac{\partial p(a_1, e_i)}{\partial e_i}.$$

This holds if and only if $\frac{\partial p(a_1, e_i)}{\partial e_i} = 0$, which would imply $p(a, e_i)$ is a constant, or

$$\delta(a_1, a_0) = 1 - \frac{y(a_0, a_0)}{2y(a_0, a_1)}.$$

Therefore, in this example ex ante investments e are surplus efficient given matching μ if, and only if, $\delta(a, a')y(a, a')$ coincide with payoffs under fully transferable utility. Otherwise by (GDD) $\delta(a_1, a_0)$ is "too small", inducing over-investment in attribute a_0. Then a social planner can change investment levels without changing the matching pattern μ and induce higher aggregate surplus. This cannot induce a Pareto improvement, however.

The failure to induce efficient investments despite ex post surplus efficiency of equilibrium payoffs in the example is due to the failure of the equal treatment property in the example. To see this suppose that the equal treatment property holds, i.e. $v(a_i) = \phi(a_i, a_j, u_j^*)$ for all $a_j \in \mu(a_i)$ for all $a_i \in A$. Denote the joint payoff in a match of attributes a and a' by

$$\hat{y}(a, a') = v(a) + v(a').$$

Then $v(a_i) = \hat{y}(a_{i-1}, a_i) - v(a_{i-1})$ for $a_i \in A_i^c$ and $a_{i-1} \in A_{i-1}^c \cap \mu(a_i)$ for distances $i = 1, .., n$ in component c of the graph G. Moreover, $v(a_0)$ with $a_0 \in A_0^c$ solve

$$v(a_0) = \hat{y}(a_0, a_0') - v(a_0') \text{ for all } a_0, a_0' \in A_0^c \text{ with } a_0 \in \mu(a_0').$$

Then the condition in Proposition 2 is satisfied if, and only if, $y(a, a') = \hat{y}(a, a')$ for all $a, a' \in A$ such that $a' \in \mu(a)$. Note that this result is robust to a perturbation of the investment technology $p(a, e_i)$. This means that if an equilibrium assignment μ is static and the equal treatment property holds for equilibrium payoffs u^*, then equilibrium investments coincide with the ones chosen by a surplus maximizing social planner for any investment technology $p(a, e_i)$ if, and only if, equilibrium payoffs u^* are surplus efficient ex post, i.e. $y(a, a') = v(a) + v(a')$ for all $a, a' \in A$ with $a' \in \mu(a)$. This means that whenever the equal treatment property holds, a social planner who cannot alter the sharing of surplus nor the match, cannot increase aggregate surplus by changing investments.

Corollary 1. *Suppose an equilibrium assignment μ is static and the equal treatment property holds for equilibrium payoffs u^*. Then investments e^* are surplus efficient if, and only if, equilibrium payoffs u^* maximize joint surplus in each match.*

The following statement gives a relation between the primitives in form of the degree of utility transferability and the equal treatment property for equilibrium payoffs, details are in the appendix.

Proposition 3 (Equal Treatment Property)**.** *The equal treatment property holds in a matching market equilibrium (μ, u^*) if $\phi(a, a', u)$ is continuous and differentiable in u and for all $a, a' \in A$ for $u \in [0, \phi(a, a\prime, 0)$*

$$0 < \frac{\partial \phi(a, a', u)}{\partial u} < -\infty.$$

The condition in the proposition implies that for any match (a, a') and given some feasible sharing of surplus, marginally increasing the surplus of one agent marginally decreases the surplus of the other agent, independently of whether this decreases or increases joint surplus. Note that this property is implied if partners in match can exchange utility at a bounded, positive rate for every feasible division of surplus.

That is, if utility is transferable at the margin at any feasible surplus sharing, a matching market equilibrium will have the equal treatment property, and equilibrium investments will be surplus efficient constrained on the match μ and partners' joint surpluses implied by equilibrium payoffs u^*.

3.2 Non-static Assignments

Focus now on equilibrium assignments μ that are not static and contain a component c satisfying case (iii) of Fact 1. A change of investments e and the subsequent changes in $q(a, e)$ trigger a change in the equilibrium assignment μ, since there is no cycle in c to adjust to balance any excess or shortfall of attributes. Then the marginal benefit from investment may jump at e^*. This is because changing investment will decrease the measure of some matches and increase the one of some other. Increasing or decreasing investment will typically affect different kind of matches, however. Surplus efficiency requires that the social marginal cost of investment lies between the different social marginal returns of increasing or decreasing investment. Hence, there will be a set of efficient investment levels.

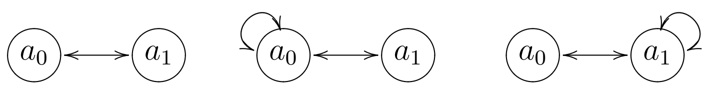

Figure 4: Graphs for an assignment μ that is not static (left), and for the corresponding assignments $\underline{\mu}$ (middle) and $\overline{\mu}$ (right).

For instance, let $A = \{a_0, a_1\}$ and $y(a_1, a_1) > y(a_0, a_1) > y(a_0, a_0)$. Suppose that $\mu(a_0) = a_1$ and $\mu(a_1) = a_0$ for equilibrium investments e^*. Let $p(a_1, e_i)$ increase in e_i. Then decreasing investments generates assignment $\underline{\mu}$ with $\underline{\mu}(a_0) = \{a_0; a_1\}$ and $\underline{\mu}(a_1) = a_0$. Increasing investments generates $\overline{\mu}$ with $\overline{\mu}(a_0) = a_1$ and $\overline{\mu}\{a_0; a_1\} = a_1$. Figure 4 shows these assignments. Using (11) this implies that investments e^* defined by $q(a_0, e^*) = q(a_1, e^*)$

are surplus efficient if, and only if,

$$[y(a_1,a_1)-y(a_0,a_1)]\frac{\partial p(a_1,e_i^*)}{\partial e_i} \leq \frac{\partial c(e_i^*,\theta_i)}{\partial e_i} \leq [y(a_0,a_1)-y(a_0,a_0)]\frac{\partial p(a_1,e_i^*)}{\partial e_i}. \tag{13}$$

This condition is satisfied by a variety of investment choices. To check whether equilibrium investments e^* satisfy the above condition, recall that e^* is given by $\sum_{a\in A} v(a)\partial p(a,e_i^*)/\partial e_i = \partial c(e_i^*,\theta_i)/\partial e_i$ for each $i \in I$. This and (13) imply that equilibrium investments are efficient if, and only if,

$$y(a_1,a_1)-y(a_0,a_1) \leq v(a_1) - v(a_0) \leq y(a_0,a_1)-y(a_0,a_0).$$

Since $q(a_0,e^*) = q(a_1,e^*)$ is a stable match, equilibrium payoffs $v(a)$ have to satisfy $v(a_0) \geq \underline{v}(a_0)$ and $v(a_1) \geq \overline{v}(a_1)$ where $\underline{v}(a)$ and $\overline{v}(a)$, denote the equilibrium payoffs for attribute a_0 and a_1 in assignments $\underline{\mu}$ and $\overline{\mu}$. Therefore

$$\overline{v}(a_1) - \overline{v}(a_0) \leq v(a_1) - v(a_0) \leq \underline{v}(a_1) - \underline{v}(a_0).$$

Hence, investments e^* such that $q(a_0,e^*) = q(a_1,e^*)$ are surplus efficient iff

$$\begin{aligned} &y(a_1,a_1)-y(a_0,a_1) \leq \overline{v}(a_1) - \overline{v}(a_0) \text{ and} \\ &\underline{v}(a_1) - \underline{v}(a_0) \leq y(a_0,a_1)-y(a_0,a_0). \end{aligned}$$

If the equal treatment property holds for assignments $\underline{\mu}$ and $\overline{\mu}$ the above condition is clearly satisfied whenever $\phi(a_i,a_j,\underline{v}(a_j)) = y(a_i,a_j) - \underline{v}(a_j)$ and $\phi(a_i,a_j,\overline{v}(a_j)) = y(a_i,a_j) - \overline{v}(a_j)$ for all $a_i,a_j \in A$. That is, under equal treatment investments are surplus efficient if equilibrium payoffs maximize the joint surplus in each match in assignments $\underline{\mu}$ and $\overline{\mu}$.

Again, surplus efficiency of payoffs $\underline{v}(.)$ and $\overline{v}(.)$ is necessary, but not necessarily sufficient for surplus efficient investments. See the appendix for an counterexample where payoff maximizes joint surplus ex post, but the equal treatment property does not hold.

The logic in this simple example extends to more general settings as stated in the following proposition, the details are in the appendix. For each type θ denote by $\overline{\mu}_\theta$ ($\underline{\mu}_\theta$) the equilibrium assignment that arises if all agents of type θ increase (decrease) their investment e_i.

Proposition 4. *Suppose an equilibrium assignment μ is not static. If the equal treatment property holds for payoffs for all equilibrium assignments $\underline{\mu}_\theta$ and $\overline{\mu}_\theta$, then surplus efficiency of equilibrium payoffs associated to the equilibrium assignments implies that equilibrium investments are surplus efficient for any investment technology.*

The following theorem summarizes Propositions 2, 3, 4, and Corollary 1, and gives the main result of this paper.

Theorem 1 (Surplus Efficiency of Ex Ante Investments)**.** *Suppose that utility is sufficiently transferable in the sense that for all $a, a' \in A$ $\phi(a, a', u)$ is differentiable in u and*

$$0 > \frac{\partial \phi(a, a', u)}{\partial u} > -\infty \text{ for } u \in [0, \phi(a, a', 0)].$$

Then equilibrium investments are constrained surplus efficient, so that a social planner cannot increase aggregate surplus by changing only investments.

4 Discussion and Conclusion

This paper has shown that competition in large matching markets induces ex ante investments that are surplus efficient constrained on the equilibrium assignment for reasonably general investment technologies if, and only if, equilibrium payoffs in all matches coincide with payoffs under fully transferable utility. Otherwise a social planner could increase aggregate surplus by marginally adjusting individual investments and forcing marginally different sharing of surplus in matches, while maintaining the equilibrium assignment.

Moreover, ex post surplus efficiency of equilibrium payoffs is a sufficient condition for surplus efficient investment only if equilibrium payoffs satisfy the equal treatment property. This property holds if utility is transferable enough to enable partners in any match to transfer utility at a finite, strictly positive rate for any division of surplus. Indeed, this induces sufficient flexibility in market payoffs to enable the accurate pricing of all externalities generated by an agent's investment choice given the limitations in utility transferability. In this case a social planner cannot increase aggregate surplus by changing individual investments alone, even when payoffs are not surplus efficient ex post.

One particular case that necessarily satisfies equal treatment in equilibrium is when the matching equilibrium takes the form of full segregation generating only homogenous matches. In this case ex post surplus efficiency of payoffs implies surplus efficiency of investments conditional on the equilibrium assignment. Whenever the equilibrium assignment involves heterogeneous matches, however, equal treatment need not be the case, in particular when payoffs have to be shared according to fixed ratios, for instance because of renegotiations.

Many of the efficiency results derived above are conditional on the equilibrium assignment of agents. While leaving open the possibility of coordination failure as a consequence of rational expectation (explored e.g. by Bhaskar and Hopkins, 2011, in a two-sided framework), this has some interesting implication when the matching of individuals is used as a policy tool, for instance in form of affirmative action or team formation. Such policies will therefore yield constrained efficient investments, conditional on the matching that is imposed and the degree of transferability. Indeed, Gall et al. (2009) examines the effects of such policies on ex ante investments and their aggregate consequences.

A Mathematical Appendix

Details for example: moral hazard in partnerships

Optimal effort choices $x_i(s)$ and $x_j(s)$ depend on s and satisfy

$$x_i(s) = (\alpha s)^{\frac{1+\alpha}{2}}((1-\alpha)(1-s))^{\frac{1-\alpha}{2}}R(a_i,a_j) \text{ and}$$
$$x_j(s) = (\alpha s)^{\frac{\alpha}{2}}((1-\alpha)(1-s))^{\frac{2-\alpha}{2}}R(a_i,a_j).$$

The sharing rule s^* that maximizes joint payoff $u_i(s)+u_j(s)$ solves

$$\max_s (\alpha s)^{\alpha}((1-\alpha)(1-s))^{1-\alpha}R(a_i,a_j)^2[1-(\alpha s^2+(1-\alpha)(1-s)^2)/2].$$

Note that the surplus maximizing sharing rule s^* is a function of α but not of $R(a, a)$, and $s^* = 1/2$ if and only if $\alpha = 1/2$. Maximal surplus in match (i, j), $y(a_i, a_j) = u_i(s^*) + u_j(s^*)$ is

$$y(a_i, a_j) = (\alpha s^*)^\alpha ((1-\alpha)(1-s^*))^{1-\alpha} \frac{2-\alpha(s^*)^2+(1-\alpha)(1-s^*)^2}{2} R(a_i, a_j)^2.$$

Setting $u_i(\hat{s}) = u_j(\hat{s})$ implies $\hat{s} = 1/2$ if $\alpha = 1/2$ and otherwise

$$\hat{s} = \frac{1 + \alpha - \sqrt{(1 + \alpha)(2 - \alpha)}}{2\alpha - 1}.$$

Indeed $\hat{s} = 1/2 = s^*$ for $\alpha = 1/2$. Otherwise $s^* > \hat{s}$ since the sum $u_i(s)+u_j(s)$ strictly increases in s at $s = \hat{s}$.

Finally, to check the condition in Proposition 1, compute the difference in $y(a, a)/2 = (u_i(s^*) + u_j(s^*))/2$ and $u_i(\hat{s}) = u_j(\hat{s}) = \phi(a, a, u_a^*)$:

$$\frac{y(a, a)}{2} - \phi(a, a, u_a^*) = \frac{y(a, a)}{2} \left(\frac{\hat{s}}{s^*}\right)^\alpha \left(\frac{1 - \hat{s}}{1 - s^*}\right)^{1-\alpha} \frac{2 - \alpha\hat{s}^2 - (1 - \alpha)(1 - \hat{s})^2}{2 - \alpha(s^*)^2 - (1 - \alpha)(1 - s^*)^2}.$$

Since neither s^* nor $\hat{s}$ depend on $R(a, a)$ the difference is a constant fraction of $R(a, a)^2$.

Note that a sufficient condition for full segregation in equilibrium for any sharing of surplus is that the maximum utility attribute a can obtain when matching with $a' < a$ falls short of sharing the surplus in a (a, a) match, that is, if for all $a, a' \in A$ with $a' < a$

$$\left(\frac{\overline{s}}{\hat{s}}\right)^{1+\alpha} \left(\frac{1 - \overline{s}}{1 - \hat{s}}\right)^{1-\alpha} \frac{2 - \alpha\overline{s}}{2 - \alpha\hat{s}} < \left(\frac{R(a, a)}{R(a, a')}\right)^2, \tag{14}$$

where $\overline{s} = \arg\max_s u_i(s)$,

$$\overline{s} = \frac{4 + 2\alpha + \alpha^2 - \sqrt{(2 - \alpha)(8 - 6\alpha^2 - \alpha^3)}}{6\alpha}.$$

Note that (14) holds whenever the additional revenue generated by another high attribute $R(a, a) - R(a, a')$ is sufficiently great for all attributes.

Edges and Vertices

A necessary condition for a unique matching equilibrium $\mu(.)$ is that $|E^c| \leq |A^c|$. Otherwise the system of equations

$$q(a, e) = \sum_{a' \in \mu(a)} \rho(a, a') + 2\rho(a, a) \text{ for } a \in A^c \tag{15}$$

has a solution ρ_1 such that $\rho_1(a, a') = 0$ for some $a, a' \in A^c$. Since all matches defined by μ cannot be blocked by other matches as μ is a matching equilibrium, the assignment defined by ρ_1 must be a matching equilibrium as well, with $\mu_1 \neq \mu$. Hence, $|E^c| \leq |A^c|$ is a necessary condition for uniqueness of μ.

Suppose a matching equilibrium μ such that $|E^c| > |A^c|$. Then the maximal surplus choosing ρ satisfying (15) can be achieved by a choice of ρ with $\rho(a, a') = 0$ for some $a, a' \in A^c$. Otherwise ρ can still be changed such that surplus weakly increases, since if there is a change of ρ that strictly decreases total surplus there must an opposite change that increases total surplus.

Finally, suppose that a matching equilibrium satisfies the equal treatment property and has $|E^c| > |A^c|$. Then choosing ρ such that $\rho(a, a') = 0$ for some $a, a' \in A^c$ will not alter payoffs since by the equal treatment property all attributes are indifferent between all their matches.

Proof of Fact 1

(i) Let a component c of G induced by an equilibrium assignment μ contain some a such that $a \in \mu(a)$. Suppose that c also contains a cycle. Then $|E^c| > |A^c|$, since a cycle has as many edges as vertices. The same argument can be applied to the case of c containing some $a' \neq a$ with $a' \in \mu(a')$ thus establishing the first statement.

(ii) If $a \notin \mu(a)$ for all $a \in A^c$ and c can be a chain or cycle. Suppose the latter then $|\mu(a)| = 2$, since otherwise the number of edges would exceed the number of vertices in c.

(iii) Suppose $a \notin \mu(a)$ for all $a \in A^c$ and c is a chain, that is c contains some terminal node a, i.e. $\mu(a) = a'$ with $a \neq a'$. This implies that $|A^c| > |E^c|$.

Proof of Proposition 3

Suppose the condition in the proposition holds, but equal treatment in equilibrium does not. Then in equilibrium there is an attribute a_i with $a_k, a_j \in \mu(a_i)$ such that $\phi(a_i, a_j, u_j^*) > \phi(a_i, a_k, u_k^*)$. But then an agent with attribute a_i who is matched to an agent with an attribute a_k, and an agent with an attribute a_j who is matched to an agent with attribute a_i, find it both strictly profitable to match together if there are payoffs $\phi(a_i, a_k, u_k^*) + \epsilon_i$ with $\epsilon_i > 0$ for the one with a_i and $u_j^* + \epsilon_j$ with $\epsilon_j > 0$ for the one with a_j, such that

$$\phi(a_i, a_k, u_k^*) + \epsilon_i \leq \phi(a_i, a_j, u_j^* + \epsilon_j).$$

Since $\phi(a_i, a_j, u_j^*) > \phi(a_i, a_k, u_k^*)$ by assumption, condition (A) is ensured if the function $\phi(a_i, a_k, u)$ is continuous in u and strictly decreasing with a slope bounded away from $-\infty$. Noting that $\phi(a, a', u)$ is non-increasing in u by definition, this is implied by the condition in the proposition.

Example when the assignment is not static

Turn now to an example where deviating matchings $\underline{\mu}$ and $\overline{\mu}$ do not involve homogenous matches. For this let $A = \{0; 1; 2\}$ and assume that $y(a, a') = \sqrt{a + a'}$. Let $p(1, e_i) = p(2, e_i) = e_i/2$ and suppose $c(e_i, \theta_i) = \theta_i e^2$ and set $\theta_i = 1/2$ for all $i \in I$.

Surplus can be split equally in homogenous matches but has to be shared according to rule δ_{ij} between attributes a_i and a_j. Suppose that in a matching equilibrium $\mu(0) = 2$ and $\mu(2) = 0$, and $\mu(1) = 1$. Since investments are pinned down by equilibrium payoffs $(\delta_{02}\sqrt{2} + \sqrt{2}/2)/2 - (1 - \delta_{02})\sqrt{2}$, $\delta_{02} = 1/2 + 4/(9\sqrt{2})$ ensures that $e^* = 2/3$ which in turn ensures that $q(0, e^*) = q(2, e^*)$.

Stability requires $\delta_{02}\sqrt{2} \geq 1$, which is true. Suppose moreover that $\delta_{01} > \sqrt{2}/2$ but $(1 - \delta_{02})\sqrt{2} > 1 - \delta_{01}$, and that $(1 - \delta_{12})\sqrt{3} > \sqrt{2}/2$ but $\delta_{02}\sqrt{2} > \delta_{12}\sqrt{3} \geq 1$ (that is, 1 agents prefer 0 and 2 agents over 1 agents but 0 and 2 prefer each other).

These assumptions imply also that increasing investment yields additional $(1, 2)$ matches, decreasing investment yields additional $(0, 1)$ matches. Hence, investments are surplus efficient if, and only if

$$[y(0,2) - y(1,2)] \frac{\partial p(0, e_i^*)}{\partial e_i} + [y(1,2) - y(1,1)] \frac{\partial p(2, e_i^*)}{\partial e_i} \leq 2/3$$
$$2/3 \leq [y(0,1) - y(1,1)] \frac{\partial p(0, e_i^*)}{\partial e_i} + [y(0,2) - y(0,1)] \frac{\partial p(2, e_i^*)}{\partial e_i}.$$

Using the functional forms defined above reveals that the second condition fails. Hence, there is over-investment, in the sense that decreasing investments e_i will increase aggregate surplus in the new matching market equilibrium corresponding to the decreased investments.

Proof of Proposition 4

Denote by $\underline{G}$ and $\overline{G}$ the graphs associated to $\underline{\mu}_\theta$ and $\overline{\mu}_\theta$, and their set of connected components by $\underline{C}$ and $\overline{C}$. Equilibrium investments e^* are surplus efficient if for each $i \in I$ of type θ

$$\frac{\partial c(e_i^*, \theta)}{\partial e_i} \in \left[\sum_{c \in \underline{C}} \underline{\sigma}^c, \sum_{c \in \overline{C}} \overline{\sigma}^c \right], \tag{16}$$

using the expression derived above for σ^c for each connected component of the graphs $\underline{G}$ and $\overline{G}$ (note both graphs $\underline{G}$ and $\overline{G}$ will be typically static). Since the cost is strictly convex $\sum_{c \in \underline{C}} \underline{\sigma}^c < \sum_{c \in \overline{C}} \overline{\sigma}^c$.

Stability of the assignment μ generating a cycle in component c implies that for all $a \in A$

$$v(a) \geq \phi(a, a', v(a')) \text{ for all } a' \in A.$$

This implies in particular that for all agents i of type θ

$$\frac{\partial c(e_i^*, \theta)}{\partial e_i} \in \left[\sum_{c \in \underline{C}} \sum_{a \in c} \underline{v}(a) \frac{\partial p(a, e_i^*)}{\partial e_i}, \sum_{c \in \overline{C}} \sum_{a \in c} \overline{v}(a) \frac{\partial p(a, e_i^*)}{\partial e_i} \right], \tag{17}$$

where $\underline{v}(a)$ and $\overline{v}(a)$ denote the equilibrium payoffs in matchings $\underline{\mu}_\theta$ and $\overline{\mu}_\theta$ as defined above. Since investment cost is convex, $\sum_{a \in A} \underline{v}(a) \frac{\partial p(a, e_i^*)}{\partial e_i} < \sum_{a \in A} \overline{v}(a) \frac{\partial p(a, e_i^*)}{\partial e_i}$.

Efficiency of investments in any matching equilibrium that is not static therefore requires that

$$\sum_{c \in \underline{C}} \underline{\sigma}^c \leq \sum_{a \in A} \underline{v}(a) \frac{\partial p(a, e_i^*)}{\partial e_i} \text{ and } \sum_{a \in A} \overline{v}(a) \frac{\partial p(a, e_i^*)}{\partial e_i} \leq \sum_{c \in \overline{C}} \overline{\sigma}^c. \tag{18}$$

Note that by the arguments above lower and upper bounds coincide if equilibrium payoffs coincide with those when utility is perfectly transferable. Otherwise the investment technology has to exactly offset any distortions. Corollary 1 implies that if the equal treatment property holds for equilibrium payoffs in matches $\underline{\mu}$ and $\overline{\mu}$ surplus efficiency of equilibrium payoffs implies that both conditions in (18) hold with equality.

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