The Role of Group Size in Group Lending

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Abstract

We explore group size in joint liability lending, primarily in the adverse selection framework with local borrower information (Ghatak 1999, 2000). We show that among homogeneous-matching contracts, a single, standardized contract charging full liability when affordable, and the maximum affordable amount otherwise, is optimal. Further, if maximum affordability is moderately high, this contract results in perfectly efficient lending if groups are large enough. However, raising group size accomplishes nothing if there is no local borrower information, showing that more is required for efficient lending than full intra-group insurance and suggesting a complementarity between group size and social capital. We show very similar results in two different settings, *ex ante* and *ex post* moral hazard – though the latter framework provides an exception in that raising group size improves efficiency even in the absence of social capital. Returning to the baseline framework, we take a step toward modeling drawbacks of larger groups, showing that if information deteriorates sufficiently with group size, an interior group size does better than either extreme. Simulations suggest that most of the efficiency gains from larger groups are realized in group sizes below ten, and that outreach and efficiency can increase discontinuously when a moderate group size threshold is crossed.

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1 Introduction

Microcredit has been seen by many as one of the great recent breakthroughs in development. Formal credit access has been extended to tens of millions of relatively poor households worldwide over the past few decades. Economic impacts are not clear but seem likely to be positive overall, given that most of this lending now occurs without subsidies.

One of the mysteries of microcredit is how unsubsidized lending has become feasible in markets where use of collateral is not possible.[1] Significant economic research has looked for clues, especially in the novel practices of microlenders. Hence, much attention has been given to group lending, where microlenders lend to individuals as members of groups that meet together for repayment and, often, bear liability for each other's loans.

Group lending is clearly not the only answer.[2] A number of prominent microlenders have never used group lending, e.g. BRI, and others have transitioned away from it at least to some degree, e.g. Grameen. These facts raise interesting questions about whether group lending, and particularly joint liability lending, remains optimal relative to other lending formats – if it ever was.

Along this line, a growing theoretical literature seeks to understand under what conditions, if any, joint liability lending can improve lending outcomes relative to other forms of lending. The results are typically nuanced – joint liability lending can improve outcomes in some dimensions and/or under certain conditions.[3]

Yet, while joint liability is not the only answer, it still seems likely to be a substantial

[1] According to the 2006 Nobel Peace Prize press release: "Loans to poor people without any financial security had appeared to be an impossible idea" (www.nobelprize.org).

[2] Armendariz and Morduch (2000) discuss other candidates for increased feasibility of microcredit.

[3] For example, in Baland et al. (2013), joint liability lending is compared to individual lending; in some cases, it can do worse in outreach to poor borrowers, but better in borrower welfare. In Ahlin and Waters (2013), joint liability lending is compared to dynamic lending, and it can dominate when agents have worse non-borrowing options. In de Quidt et al. (2012a), joint liability lending is compared to group lending without joint liability and to individual lending, and it can dominate for intermediate levels of social capital.

This nuance is present also in the early work on group lending. Stiglitz (1990) shows that joint liability lending raises borrower risk relative to individual lending; but with sufficient social capital, it can improve risk-pricing and allow for greater loan size. Besley and Coate (1995) shows that joint liability lending raises repayment in some states of the world but lowers it in others, with ambiguous net implications unless social sanctions are strong enough.

part of it – and not only historically. There appears to be no statistical evidence of a trend away from joint liability lending, and de Quidt et al. (2012a,b) document a majority of loans being given through some form of joint liability in a recent worldwide database of microlenders. Thus, while understanding joint liability and its role in improving lending outcomes is certainly not the only worthwhile line of inquiry in understanding the feasibility of microlending, it seems as relevant as ever.

However, a significant gap exists in the current theoretical literature on joint-liability based microcredit: it typically focuses on groups of size two. (Exceptions and related literature are discussed below.) This is undoubtedly for reasons of simplicity and/or tractability rather than empirical relevance, since groups often have between four and ten members, and rarely two. The goal of this paper is to relax the assumption of two-member groups in analyzing optimal group contracts and lending outcomes, and thus to explore the role of group size. We do so in a range of theoretical contexts, and with varying degrees of borrower social capital.

The primary analysis is in the context of adverse selection, specifically the framework of Ghatak (1999, 2000), based on Stiglitz and Weiss (1981). Borrowers have projects with identical mean returns, all worthwhile to fund. The projects differ in risk levels, with risk unobservable by the zero-profit lender but common knowledge among borrowers. Under standard individual loans with limited liability, safer borrowers pay more in expectation than risky – they succeed, and thus repay their loans, more often. The need to pay this cross-subsidy if they borrow can keep safer borrowers out of the market, leading to partial or complete market breakdown.

Ghatak (1999, 2000) and Gangopadhyay et al. (2005) have shown in this context that joint liability lending to groups of size two can, but may not, restore the market to efficiency. The efficiency gains rely on the incentives for borrowers to sort homogeneously and the improved risk-pricing that results, both stemming from the joint liability contract.[4]

[4] Evidence for homogeneous matching and improved risk-pricing can be found in Ahlin (2009).

In this context, we consider lending to groups of size $n \geq 2$. We begin by deriving an optimal contract subject to several constraints, including limited liability; monotonicity, as in Innes (1990) and Gangopadhyay et al. (2005), which guarantees that borrowers cannot be liable for more than their fellow group members' loan values; and equilibrium homogeneous matching, since joint liability contracts by themselves do not guarantee this.

Define an affordable full liability contract ("AFLC") as the contract that requires the group to repay its full obligation unless too many members have failed, in which case each successful borrower pays the maximum amount affordable by all. That is, an AFLC involves full liability to the extent allowed by limited liability. We show three main results about contract form. First, an AFLC uniquely induces homogeneous matching. Second, assuming a single contract is offered to all, an AFLC is optimal. Third, no menu of contracts can achieve higher efficiency than a single AFLC offered to all – i.e., pooling is without loss of generality.

These results rationalize a contract form that is arguably the canonical form of group lending, given that many group-based microlenders make borrowers fully liable for all group loans. The intuition here is that among monotonic contracts, full liability maximizes borrower payments in states of the world with more failures, which are experienced disproportionately by risky borrowers. Thus, by shifting the repayment burden toward risky borrowers as much as possible, full liability minimizes the cross-subsidy risky borrowers receive from safe borrowers, i.e. it most accurately prices for risk.

Knowing the optimal contract, how does efficiency vary with group size? Under an affordability condition, we show that fully efficient lending – complete outreach and maximal borrower surplus – is always achievable by group lending with large enough groups. The affordability condition is quite plausible, because it does not depend on a borrower being able to bail out the entire group, but only his share of failed borrowers at *mean* failure rates.

This main result is possible because under an AFLC, increasing group size sends toward one the probability that the group will be able to repay its entire obligation – essentially, it provides asymptotically full insurance to the group. However, within-group insurance is not sufficient. We show that group size is irrelevant if an AFLC is used but there is no local information, i.e. random matching, even though large groups can attain asymptotically full insurance under random matching.

The reason behind these results is that there are two potential sources of the cross-subsidy that works against efficient lending: within-group and between-group. Group repayment with certainty eliminates cross-subsidies between groups, regardless of matching patterns, by ensuring all groups pay their share. However, it does nothing about within-group cross-subsidies. Under random matching, a group's obligations are borne more heavily by the safer borrowers within the group, and this potential within-group cross-subsidy eliminates any gain from group lending.

By contrast, the AFLC with local information induces homogeneous matching – eliminating the within-group cross-subsidy – and, with sufficiently large groups, causes the probability of full group repayment to approach one – asymptotically eliminating the between-group cross-subsidy. Asymptotically all borrowers pay the same and none are excluded. Thus, increasing group size is an effective tool for improving risk-pricing and achieving efficient lending, but only given sufficient social assets (local borrower information).

How widely applicable are these results? We next show they apply in two quite different settings, an *ex ante* moral hazard setting as in Stiglitz and Weiss (1981) and Stiglitz (1990); and an *ex post* moral hazard setting as in Baland et al. (2013; "BSW").

The Stiglitz setting features unobserved project choice, where limited liability skews the borrowers' incentives toward taking on excess, inefficient risk. In this context, under the same key affordability condition, fully efficient lending is also achievable with an AFLC and large enough groups. While intragroup insurance is critical to this result, it also depends critically on borrowers being able to decide cooperatively on projects that maximize total group payoffs. If instead borrowers act non-cooperatively, i.e. can unilaterally deviate from the group-optimal project choice, then group lending offers no improvement over individual lending regardless of group size. So again, group size is an effective tool, but only in conjunction with a social asset.

In the BSW setting, borrowers can freely walk away from loans, but the lender can impose non-pecuniary sanctions. Sanctions are critical for repayment incentives, but they also entail efficiency losses. Unlike most previous work, BSW analyze the role of group size; however, they focus on local properties and non-monotonicities. Our work complements theirs by analyzing outcomes when group size can be set freely.

BSW show that in the absence of social sanctions, group lending achieves strictly lower outreach than individual lending, but can sometimes provide higher payoffs to those who do borrow. They also show that borrower welfare can sometimes be raised by using smaller groups. We show that almost every borrower that can be reached by individual lending can be reached by group lending with large enough groups – and with strictly higher payoffs. We also show that while welfare can sometimes be increased by using a smaller group size, it can always be raised even more by using a larger group size. Thus, again group lending with adequately large groups nearly completely dominates individual lending – interestingly here, unlike in the other contexts, without the assumption of any social capital.

BSW also allow for social sanctions. While the availability of social sanctions enhances outreach and efficiency of group lending, as they show, even with arbitrarily large social sanctions, individual lending may still achieve better outreach than group lending. However, this is with fixed group size. Here we show that any positive amount of social sanctions means group lending dominates individual lending in both outreach and borrower payoffs for large enough groups; and social sanctions above a fixed level guarantee complete outreach and borrower payoffs arbitrarily close to efficiency for large enough groups.

Together, these results show that group size combined with some social asset can be an effective tool in achieving efficient lending. There is also an interesting difference in that group size without any explicit social asset is ineffective in two environments but quite valuable in the other – suggesting a potential direction for future empirical work exploring complementarities between group size and social capital.

The results on group size so far have been asymptotic. The question remains whether the mechanisms highlighted are relevant for reasonable group sizes. We address this question in two ways. First, we return to the baseline adverse selection model and assume that local information deteriorates as group size increases, and with it homogeneous matching. Not surprisingly, assuming that local information vanishes in the limit, a moderate group size attains higher efficiency than either extreme.

Second, we simulate the model using what we argue are reasonable parameter estimates. The simulations show that the asymptotic arguments are applicable at reasonable group sizes, and even when the affordability assumption is not met: most of the efficiency gains to large groups occur for group sizes below 10. We also observe a "discontinuity" in efficiency as group size increases, with efficiency jumping substantially as a typically moderate threshold is reached – in our simulations, in the 2-5 range. Finally, we see that higher project returns relax affordability constraints and can substitute for larger group size – a prediction that future empirical work could explore. Overall, the results seem to help rationalize observed group sizes as not just incidental but perhaps critical to the effective use of group lending.

Section 2 discusses related literature. Section 3 sets out the model and contract restrictions, while Section 4 provides results on optimal contract form. Section 5 shows the effect of group size in the baseline setting, while Section 6 analyzes group size in two other settings. Section 7 takes up limits to group size theoretically and presents simulation results, and Section 8 concludes. Proofs are in the Appendix.

2 Relation to the Literature

A number of this paper's broad ideas about group size can be found elsewhere. Some of the first to discuss group size, though without formal modeling, are Ghatak and Guinnane (1999) and Ghatak (2000). They argue that a potential advantage of larger groups is the greater likelihood the group can repay its loans, and also point out that a likely disadvantage of larger groups is deteriorating social information and cooperation. Though in different contexts than ours, Diamond (1984), Laux (2001), and Conning (2005) provide formal models showing that large groups can be used to improve efficiency, and even asymptotically attain the first-best, again due to elimination of group-level risk.[5] Conning (2005), the closest to our paper, also shows that groups that are too large may not be able to enforce the cooperative agreements necessary for group repayment.[6]

Yet, the literature does not present a fully settled view on group size. Baland et al. (2013; "BSW") show that sometimes a smaller group size can raise efficiency, in a setting similar to Diamond (1984) and Laux (2001). And, in the only formal treatment of group size in our focal context of adverse selection (Ghatak, 1999, Appendix A.1), group size is irrelevant.[7] The point is that the literature could seemingly benefit from a detailed analysis of group size that draws these different results together, in the adverse selection context and beyond.

Our paper contributes to the literature on group size in lending in a number of ways. With the already noted exception of Ghatak (1999), it is the first analysis of group size in the adverse selection setting.[8] It is not obvious why group size should have any impact in this setting where there are no penalties for default, incentives are irrelevant, and full insurance within the group has no obvious benefit. In this context, we derive an optimal contract (which explains the difference from previous results in this context) and demonstrate homogeneous matching. We also show that the basic logic from the literature – that large

[5]Che (2002) also shows that larger groups bring about greater efficiency. The logic is quite different – it is related to the fact that the punishment, in a repeated game where all fellow group members shirk in retaliation for one member shirking, is stronger the more group members there are.

[6]One difference is that Conning does not impose monotonicity on the contract, and so analyzes contracts where a borrower gets positive payoffs only if *all* projects in his group succeed.

[7]That is, any impact of a change in group size can be accomplished by a change in the degree of joint liability. However, the analysis is mainly a robustness check of homogeneous matching, not a full investigation of group size; accordingly, attention is restricted to a linear form of the joint liability contract.

[8]The remaining literature on group lending under adverse selection analyzes groups of size two. This includes Ghatak (2000), Van Tassel (1999), Laffont and N'Guessan (2000), Laffont (2003), and Armendariz and Gollier (2000). Gangopadhyay et al. (2005) extends the Ghatak model to include a monotonicity constraint, still with groups of size two.

groups can repay with near certainty – is an important but not sufficient aspect of increasing group size in this setting. The presence of a social asset – local borrower information – is necessary for large groups to have any impact. In addition, we find that affordability is also critical in this context where there is heterogeneity in project returns, but that large groups make affordability at mean failure rates the determining factor, rather than affordability in tail events.

The results here also clear up the seeming counterexample provided by the BSW result that, even without any deterioration of group social capital as group size increases, welfare can sometimes be improved by a smaller group size. We show that it is also true that welfare can be improved *even more* by a larger group size. This frames the BSW result as about local monotonicity rather than the general tendency toward efficient lending as group size increases.

We also add to the literature an exploration of how the effect of group size varies with social capital.[9] We find that in two settings – adverse selection and *ex ante* moral hazard – some social asset is needed for group size to have any effect, while in a third setting – *ex post* moral hazard – group size can be a useful tool even without a social asset. This introduces to the literature an idea with empirical applications, that there can exist a complementarity between group size and social capital, and that it may vary depending on the key impediment to lending.

To our knowledge, our paper is also the first to show via simulation that the beneficial effects of larger groups are often realized at reasonable group levels, and that efficiency can jump dramatically once a key group size is reached. These results are a step toward rationalizing observed group sizes, and they suggest that the asymptotic arguments of the other theoretical literature may also have practical relevance.

[9] Also relevant to this theme are the results of Conning (2005) and BSW. In an *ex ante* moral hazard effort-provision context, Conning shows that the ability of a group to act cooperatively is necessary for group lending to have any effect, and given this ability, sufficiently large groups attain first-best lending. BSW examine the role of group size with and without social sanctions; however, unlike them, we map outcomes when group size can be set freely.

Finally, we know of no other papers that establish optimality of a single, standardized contract involving full liability whenever affordable and the maximum affordable amount otherwise. This is often taken to be the canonical group contract in microcredit, and it is often assumed rather than justified. Hence, this result breaks ground in rationalizing a canonical form of group lending. It also points to the idea that screening using multiple contracts is not central to the ability of group lending to overcome adverse selection – the key is improved risk-pricing, which can be accomplished by offering a single contract.

3 Basic Model

3.1 Environment

There is a unit measure continuum of risk-neutral agents, indexed by $i \in [0,1]$. Each is endowed with no capital, one unit of labor, a subsistence option, and a project. The subsistence option requires one unit of labor only and gives expected output $\overline{u} \geq 0$. The project requires one unit each of labor and capital.

Agent i's project yields gross returns of R_i with probability p_i and yields 0 gross returns with probability $1 - p_i$. Project returns are distributed independently across agents. As in Stiglitz-Weiss (1981), assume that all projects have the same expected value:

$$p_i \cdot R_i = \overline{R}, \quad \forall i \ . \tag{A1}$$

However, the projects differ in risk: the p_i's are distributed over $[p_r, p_s]$ according to strictly positive and continuous density function $f(p)$, where $0 < p_r < p_s < 1$. Denote the expected value of $g(p)$ in the population as $\overline{g(p)}$.[10]

Agents require outside funding to carry out their projects. We consider a non-profit lender that maximizes total borrower surplus subject to earning expected gross return $\rho > 0$

[10] So, $\overline{g(p)} = \int_{p_r}^{p_s} g(p)f(p)dp$. For example, $\overline{p} = E(p) = \int_{p_r}^{p_s} pf(p)dp$ and $\overline{p^2} = E(p^2) = \int_{p_r}^{p_s} p^2 f(p)dp$.

on capital. It is assumed that

$$\overline{R} > \rho + \overline{u} \; . \tag{A2}$$

This implies that in expectation, all projects return more than the cost of their inputs, capital and labor. Thus, social surplus is monotonically increasing in outreach, i.e. the number of projects funded, and the efficient outcome occurs iff all agents borrow.

Agents' risk-types are observable to other agents, but not to the lender. Further, project returns are publicly verifiable, but only coarsely: it is costless to distinguish between $Y = 0$ (fail) and $Y > 0$ (succeed), but prohibitively costly to distinguish between different levels of $Y > 0$.[11]

There are three parameters governing rates of return: $\overline{R}$, $\overline{u}$, and ρ. We will often think in terms of an equivalent set of three parameters, ρ, $\mathcal{G}$, and $\mathcal{N}$, where

$$\mathcal{G} \equiv \frac{\overline{R}}{\rho} \qquad \textit{and} \qquad \mathcal{N} \equiv \frac{\overline{R} - \overline{u}}{\rho} \; . \tag{1}$$

$\mathcal{G}$ can be interpreted as the gross excess return to capital, and $\mathcal{N}$ as the net excess return to capital, of these agents' projects: the numerator is the return to a unit of capital in the agents' projects (gross or net of opportunity cost of labor $\overline{u}$), and the denominator is the return to a unit of capital elsewhere. It is clear that there is a one-to-one mapping between these two sets of parameters. Further, ρ, $\mathcal{G}$, and $\mathcal{N}$ can vary independently, subject only to the following restrictions inherited from restrictions on $\overline{R}$, $\overline{u}$, and ρ: $\rho > 0$, $\mathcal{G} \geq \mathcal{N}$, and the equivalent of Assumption A2 guaranteeing it is efficient to fund all projects:

$$\mathcal{N} > 1 \; . \tag{A2}$$

[11] Rai and Sjostrom (2004) consider the case where agents do observe each others' output levels in an *ex post* moral hazard environment, and show that cross-reporting mechanisms can and must be used at the optimum. Here, agents do not observe each other's output, but it may be that a cross-reporting mechanism could elicit agents to report each other's type. We rule these mechanisms out by assumption.

3.2 Contracts for n-person Groups

We restrict attention to deterministic, symmetric contracts, and discuss next three additional restrictions imposed on group contracts in the analysis.

Limited liability. We assume that agents' exposure in any financial contract is limited to project returns. Limited liability implies nothing is due from a borrower who fails. Since the lender cannot verify the level of successful output, we assume any payment due upon success must be affordable by all potential borrowers.[12] Since the safest borrowers earn the least when successful (under assumption A1), this caps payment due from any successful borrower at R_s.

Unverifiability of successful output level along with limited liability makes debt contracts the only feasible financial contracts. Consider a symmetric lending contract for borrowers in groups of fixed integer size $n \geq 2$. Such a **group contract** can be captured by n non-negative interest rates: $(r_0, r_1, ..., r_{n-1})$, where r_k is the amount due from a borrower who succeeds and k of whose fellow group members fail, $k \in \{0, 1, ..., n-1\}$.

Homogeneous matching. We consider only group contracts that uniquely induce homogeneous matching of the borrowers, i.e. matching such that borrowers within groups have identical types. This is a slight departure from previous literature, which considers only joint liability contracts, then shows that joint liability induces homogeneous matching (Ghatak 1999, 2000). This approach appears infeasible with group size $n > 2$, because joint liability no longer guarantees homogeneous matching, unlike when $n = 2$.

To see this, define a joint liability contract as a group contract that satisfies:

$$r_0 \leq r_1 \leq r_2 \leq ... \leq r_{n-1}, \qquad \textit{and} \qquad r_0 < r_{n-1} \; .$$

Thus, for a group contract to involve joint liability, the amount a borrower owes must be increasing in the number of failures in the group, strictly so at least somewhere.

[12]An alternative approach would be to assume it must be affordable only by all *equilibrium* borrowers, but this complicates off-equilibrium scenarios.

In both the baseline Ghatak model (1999, 2000) with $n = 2$ and the Ghatak extension (1999) for $n > 2$, attention is restricted to two-parameter contracts of the form $r_k = r + ck$, for $k \in \{0, 1, ..., n-1\}$. In both cases, homogeneous group formation occurs as the unique stable matching outcome iff $c > 0$, i.e. iff the two-parameter contract is a joint liability contract by the above definition.

Unfortunately, this equivalence does not generalize to arbitrary joint liability contracts when $n > 2$. For example, assume $p_s > 2/3$, and consider the contract $r_k = r$, $k \leq n-2$, and $r_{n-1} = r + \kappa$, for some $\kappa > 0$. One can show that the group payoff function under this contract is strictly submodular over $(2/3, 1)^n$ if $n \geq 3$ (over $(1/2, 1)^n$ if $n \geq 4$). This guarantees that homogeneous matching is not stable (see Ahlin, 2012, Proposition 3). The intuition here is that since the contract penalizes only extreme failure, spreading out safe types across groups raises payoffs by lowering chances of paying the penalty.

Thus in lending to groups with more than two members, the specifics of joint liability matter for the matching pattern. The point is that, to proceed like the previous literature and restrict attention to joint liability contracts would require venturing into theoretical matching territory where it appears little can be said without further assumptions (Ahlin, 2012). Instead, the approach taken will be to restrict attention to contracts that induce homogeneous matching uniquely, and seek optimal contracts from among this set.

Monotonicity. Following Innes (1990) and Gangopadhyay et al. (2005), we also impose the following monotonicity constraint on the group contract:

$$\begin{gathered} r_{n-1} \leq 2r_{n-2} \leq 3r_{n-3} \leq ... \leq (n-2)r_2 \leq (n-1)r_1 \leq nr_0 \ , \quad i.e. \\ (n-k)r_k \leq (n-k+1)r_{k-1} \, , \qquad k \in \{1, 2, ..., n-1\} \ . \end{gathered} \tag{2}$$

This guarantees that the total amount owed by the group is (weakly) increasing in the number of successes in the group: if there are $(n-k)$ successes, then $(n-k)r_k$ is owed by the borrowers in total, while if there are $(n-k+1)$ successes, then $(n-k+1)r_{k-1}$ is owed. The argument is that if it were cheaper to have more successes, group members could simply claim some of their fellow members succeeded in order to lower the total payment due from the group.[13] If $n = 2$, this constraint is equivalent to requiring no more than full joint liability on the loan, $r_1 \leq 2r_0$. When $n \geq 3$, it also requires no more than full liability.

4 Optimal Group Contracts

As a starting point, we consider only single, standardized contract offers, i.e. "pooling contracts". This may be justified since quite a few micro-lenders apparently offer a single standard contract form, not extensive menus of contracts. More importantly, the restriction to pooling contracts is without loss of generality for efficiency, as we prove formally below.

A preliminary result that will aid analysis is:

Lemma 1. *Under a monotonic contract that induces homogeneous matching, a safer borrower earns less than a riskier borrower, and the lender earns more from a safer borrower than from a riskier borrower. If an agent of type $\hat{p}$ borrows under such a contract, so do agents of types $p \in [p_r, \hat{p}]$.*

Among other things, this makes clear that safer borrowers are the harder to attract.

Given homogeneous matching, a group contract $(r_0, r_1, ..., r_{n-1})$ gives an agent of type p_i a borrowing payoff

$$\Pi_i = \overline{R} - \sum_{k=0}^{n-1} p_i^{n-k}(1-p_i)^k \binom{n-1}{k} r_k \ . \tag{3}$$

This payoff reflects the fact that borrower i pays r_k iff he succeeds – probability p_i – and k of his $n-1$ fellow group members fail – probability $p_i^{n-1-k}(1-p_i)^k\binom{n-1}{k}$. If all agents borrow, the zero-profit constraint ("ZPC") is

$$\sum_{k=0}^{n-1} \overline{p^{n-k}(1-p)^k} \binom{n-1}{k} r_k \geq \rho \ . \tag{4}$$

[13]This constraint is motivated and analyzed in the $n = 2$ context by Gangopadhyay et al. (2005). It can also be motivated as a reduced-form constraint from a costly state verification problem in which the lender only audits when a failure is reported. Since reports of success go unverified, the constraint ensures there is no incentive to overstate success.

The summed terms are r_k times the probability of getting paid r_k from a randomly selected borrower, since $\overline{p^{n-k}(1-p)^k}$ is the population average of $p_i^{n-k}(1-p_i)^k$. If not all agents borrow, the same ZPC applies with the expectation over the set of types that do borrow.

Given group size $n \geq 2$, an optimal pooling contract chooses n interest rates $(r_0, r_1, ..., r_{n-1})$ to maximize total borrower payoffs:

$$\int_{p_r}^{p_s} max\{\Pi_i, \overline{u}\} f(p_i) dp_i$$

subject to borrower limited liability ($r_k \leq R_s$), monotonicity (constraint 2), the ZPC, and unique homogeneous matching. Because every project funded produces positive social surplus (Assumption A2), social surplus is monotonically increasing in number of projects funded, i.e. in outreach. If the lender exactly breaks even, as will always be the case, borrowers obtain all the social surplus; hence, borrower surplus is also monotonically increasing in outreach. An optimal contract will therefore maximize outreach subject to the constraints.

Define a full liability contract to be a group contract satisfying, for some $r \geq 0$,

$$r_k = \frac{nr}{n-k}, \quad k \in \{0, 1, ..., n-1\} \ .$$

Since $r_k(n-k)$ is due from the group where there are $k < n$ failures, this ensures the group pays the same amount, nr, after every outcome involving at least one success in the group. Of course, one successful borrower may not be able to cover all group loans. Hence, define an **affordable full liability contract** ("AFLC") to be a group contract satisfying, for some $r \geq 0$,

$$r_k = \min\left\{\frac{nr}{n-k}, R_s\right\}, \quad k \in \{0, 1, ..., n-1\} \ . \tag{5}$$

An AFLC imposes full liability whenever it is affordable, but caps a borrower's payment at the maximal amount affordable by all borrowers, R_s.

The AFLC clearly satisfies limited liability as well as monotonicity. And, while not all joint liability contracts satisfy the homogeneous matching constraint, the AFLC does:[14]

Lemma 2. *Homogeneous group formation is the unique stable match under any affordable full liability contract with $r \in (0, R_s)$.*

This non-trivial result is obtained via simplifications focusing on two arbitrary borrowers' contributions to event probabilities.

Knowing that affordable full liability satisfies homogeneous matching, limited liability, and monotonicity, we now show that it is optimal:

Proposition 1. *Fix $n \geq 2$ and assume there exists a group contract satisfying all constraints. Then the set of optimal pooling contracts is non-empty and includes an affordable full liability contract, and only this type of contract if some but not all projects are funded.*

Thus, an AFLC is optimal, regardless of n and the distribution of types, though there may be other contracts that do just as well in the polar cases where all borrowers or no borrowers are funded.

Maximizing the degree of joint liability is optimal here for the following reason. In this context of unobserved risk-types, what damages the lending market is the lender's inability to price for risk. This leads some borrowers to cross-subsidize others – in particular, with monotonic contracts it is the safer borrowers who repay more in expectation and thus subsidize the riskier. Anticipating paying this cross-subsidy, safer borrowers may opt out of the market, leaving profitable projects unfunded. The way toward greater efficiency is thus reducing the cross-subsidy from safe to risky borrowers, i.e. better risk-pricing, since this is what will attract more (safe) borrowers and fund more worthwhile projects.[15,16] The

[14] We use the core as the equilibrium concept for matching – this corresponds to frictionless matching with side payments allowed. See Ahlin (2012) and its references for more details.

[15] Thus cross-subsidies are not just a distributional concern here, but also the source of inefficiency.

[16] Of course, perfect risk-pricing and perfect efficiency would obtain if the lender could observe risk-type. The lender would charge higher rates to riskier borrowers and lower rates to safer borrowers, earning the same in expectation from all and thus attracting all kinds.

cross-subsidy from safe to risky is minimized when as much repayment as possible is loaded onto outcomes with more failures, since these outcomes are experienced disproportionately by riskier borrowers. This is exactly what affordable full liability accomplishes.

How binding is the restriction that the lender only offers one contract, rather than a menu that screens risk-types using the shape of joint liability (as in Ghatak, 2000, and Gangopadhyay et al., 2005) and/or group size? It is not restrictive at all:

Lemma 3. *A pooling contract satisfying all constraints can achieve the same borrower surplus as any menu of group contracts satisfying all constraints.*

This result applies to screening using all available instruments, i.e. both the interest rates and the group size. The reasoning is that the lender can achieve the same surplus by offering only the safest borrower's contract from any optimal menu. Riskier borrowers get the same payoffs, since their incentive constraints bind at the optimum (or can be made to without loss), and the lender also gets the same profit since its profits and borrower payoffs move one-for-one in opposite directions.[17]

Proposition 1 and Lemma 3 are encouraging results, since many microlenders appear to offer a single, standardized product, and affordable full liability seems to be a relatively accurate description of the canonical microcredit contract.

5 The Effect of Group Size

It is known that fully efficient lending, i.e. maximal borrower surplus and complete outreach, may not be achievable under standard individual contracts, which offer a borrower one unit of capital with zero due after failure and an amount r due after success. Since the break-even interest rate under full outreach is $\rho/\overline{p}$, the condition required is that

$$\overline{R} - p_s\rho/\overline{p} \geq \overline{u} \qquad \Longleftrightarrow \qquad \mathcal{N} \geq \frac{p_s}{\overline{p}} \equiv \overline{\mathcal{N}}_1 \ (> 1) \ .$$

[17]In other words, the screening instruments are costless to use. This is the key feature of the environment that makes standard pooling non-existence arguments inapplicable.

So for $\mathcal{N} \in (1, \overline{\mathcal{N}}_1)$, fully efficient lending is not possible with individual loans.

Also known is that monotonic group lending with $n = 2$ relaxes the condition for fully efficient lending, but only partially. In particular, an affordable full liability group contract is optimal, and even when full liability is affordable, the condition for fully efficient lending is[18]

$$\mathcal{N} \geq \frac{p_s(2 - p_s)}{\overline{p(2 - p)}} \equiv \overline{\mathcal{N}}_2 \ (\in (1, \overline{\mathcal{N}}_1)).$$

Thus for $\mathcal{N} \in (1, \overline{\mathcal{N}}_2)$, fully efficient lending is not achievable with group lending and $n = 2$.

We turn next to our main question: how does efficiency vary with group size? It turns out that the complexity of the contract makes a monotonic relationship hard to establish theoretically.[19] However, the general tendency in efficiency as group size increases is clear, and this is our focus.

One further assumption is used:

$$\mathcal{G} > \frac{p_s}{p_r} \,. \tag{A3}$$

This restriction on the gross excess return will guarantee sufficient affordability of bailout payments (see below). Note that this bound is independent of group size n, and potentially quite unrestrictive. Depending on p_r and p_s, it may require that successful borrowers only be able to afford to pay for one other loan besides their own, or even half a loan.

Proposition 2. *A large enough group size n and assumption A3 guarantee maximal borrower surplus and complete outreach are achieved by an affordable full liability group contract.*

This is our main result, and it shows that fully efficient lending is attainable if groups are adequately large, provided sufficient affordability. This is true **for any $\mathcal{N} > 1$**.

For intuition, it is interesting first to ignore affordability of interest payments after success. In this case, full liability guarantees groups pay back their full obligation except when all members fail, since even one successful member can cover the entire obligation. The

[18]This comes from solving ZPC 4 and $\Pi_s \geq \overline{u}$ simultaneously, assuming $r_1 = 2r_0$.

[19]In fact, there may be no monotonic relationship, as Baland et al. (2013) find in a different context (*ex post* moral hazard). On group size in that setting, see Section 6.2.

probability of all failing goes to zero as n gets large, so asymptotically all groups fully repay and there is no cross-subsidy across groups.

However, given more reasonable affordability assumptions, full liability is possible only if the number of failures is low enough. This creates a cross-subsidy from safe to risky groups, since risky groups get out of full liability more often by more frequently failing in greater numbers. In this context, what large groups do is concentrate failure rates more tightly around the mean; and if full liability is affordable at the mean failure rate, then the contract asymptotically implements full liability, which eliminates the cross-subsidy across groups. This is why bailout affordability need not be high in large groups, and can even be lower than in 2-person groups – what matters for large groups are not tail events, but average events.

To illustrate, consider the limit case of infinitely large groups. Then if borrower i pays

$$r + r\frac{1-p_i}{p_i} = \frac{r}{p_i} \ ,$$

when he succeeds, he repays his own loan plus his share of the group's defaulters' loans (since $1-p_i$ need bailing out and p_i are available to do so). If all borrowers can afford this rate, there is no default risk and $r = \rho$. Thus all borrowers expect to pay $p_i(\rho/p_i) = \rho$, exactly the cost of capital; i.e., risk-pricing is perfect. Regarding affordability, since the amount due (ρ/p_i) increases in risk, the maximum payment R_s must be able to cover the riskiest borrower's payment, ρ/p_r; Assumption A3 guarantees exactly this.

This result highlights the importance of the insurance that a joint liability group can provide. The risk in mean borrower payoffs within a group goes to zero as groups get large; hence, larger groups provide for greater insurance, in a sense. However, the reason that eliminating group-level risk is good here is not the standard ones, like risk aversion or maintaining the lending relationship. The elimination of payoff risk at the group level is valuable here because it eliminates the cross-subsidy across groups and improves risk-pricing

– it ensures all groups can handle full liability (in the limit) and thus all pay the same, regardless of risk-type.

This logic may give the impression that large groups give efficiency solely by a Central Limit Theorem ("CLT") argument that dates at least to Diamond (1984). But this would be misleading.

Lemma 4. *Regardless of group size n, if borrowers match randomly to form groups, a group contract satisfying borrower limited liability and lender break-even can achieve no more borrower surplus than an individual loan contract satisfying the same constraints.*

This result implies that group size is a worthless tool if borrowers match randomly (which we might expect to occur if risk-type is unobservable to other borrowers). This is true even in parameter ranges where large n and an appropriate group contract cause the group repayment probability to approach one.

The point is that groups, however large, by themselves accomplish nothing; and that CLT insurance logic is not sufficient for the result we show. This is because, while sufficiently-large groups and affordable full liability can virtually eliminate the between-group cross-subsidy, under random matching they do not eliminate the within-group cross-subsidy – the full liability within a group would be borne disproportionately by the safer borrowers within that group, leaving no improvement over individual loans.

It is also informative to note that even full within-group insurance *and* homogeneous matching are not by themselves sufficient – the contract must be tailored to eliminate the between-group cross-subsidy. Consider for example the linear contract $r_k = r + kc$, for some $c > 0$. As Ghatak (1999) has shown, this contract can be replicated with a two-person joint liability contract, regardless of group size n. Thus, maximal efficiency is the same as with two-person groups, and efficiency is not always attainable by a monotonic contract. The problem here is that the contract does not eliminate the between-group cross-subsidy: the amount due from a group varies non-negligibly with number of failures regardless of n, and under monotonicity, riskier groups pay less.

Thus, Proposition 2 and Lemma 4 show that adequate insurance within a group based on sufficiently large numbers helps, but only in conjunction with other ingredients. That is, sufficient for full efficiency in this context are both adequately-large groups and an appropriate contract – to minimize the between-group cross subsidy – and homogeneous matching – to eliminate the within-group cross subsidy. Insurance by itself is not enough for increasing group size to be effective – good local information and resulting matching are also critical.

In sum, Section 5 has shown that affordable full liability with large enough groups can always achieve fully efficient lending if gross payoffs are high enough. The result is strong in that it holds for any number of types, even a continuum, and makes use of an extremely simple, standardized debt contract, which resembles those often seen in practice.

6 Group Size in Other Contexts

Given the multiplicity of contexts in which group lending is analyzed, one might wonder whether similar results hold in other settings. In this section, we explore group size in two other contexts, *ex ante* moral hazard (unobserved project choice) as in Stiglitz and Weiss (1981) and Stiglitz (1990), and *ex post* moral hazard (strategic default, limited enforcement), as in Baland et al. (2013). We find similar results.[20]

6.1 *Ex ante* Moral Hazard and Group Size

Here we use a simplified version of the moral hazard models of Stiglitz and Weiss (1981) and Stiglitz (1990). These models involve a hidden action problem under which the lender cannot contract on project choice. Coupled with limited liability, this gives a borrower the incentive to take on excess risk, since part of the risk is borne by the lender.[21]

Specifically, all borrowers here choose between two types of all-or-nothing projects to

[20] One can also show similar results under adverse selection in the De Meza and Webb (1987) framework.

[21] Another set of hidden action models focuses on suboptimal effort provision. Conning (2005) analyzes the effort-provision environment and derives similar results.

undertake with the borrowed capital, safe and risky, with $0 < p_r < p_s < 1$,

$$p_s R_s \equiv \overline{R}_s > \overline{R}_r \equiv p_r R_r \ ,$$

but $R_s < R_r$. The risky project represents an inefficient gamble, paying off more when it succeeds but returning less on average. We also assume that the safe project pays more than the outside option, i.e $\overline{R}_s - \rho > \overline{u}$. Hence, the efficient lending outcome is safe project choice, which is also the outcome of a self-financed entrepreneur or a market with perfect information. However, given individual loan with interest rate r and limited liability, if the interest rate is high enough the borrower will prefer the risky project to the safe:

$$\overline{R}_r - rp_r > \overline{R}_s - rp_s \qquad \Longleftrightarrow \qquad r > \frac{\overline{R}_s - \overline{R}_r}{p_s - p_r} \ .$$

When can efficiency be attained by standard individual loan contracts? The break-even interest rate in that case would be $r = \rho/p_s$. The safe project is preferred to the outside option at this rate, but preferred to the risky project iff[22]

$$r = \frac{\rho}{p_s} \leq \frac{\overline{R}_s - \overline{R}_r}{p_s - p_r} \ , \qquad i.e. \ , \qquad \mathcal{N} \equiv \frac{\overline{R}_s - \overline{R}_r}{\rho} \geq 1 - \frac{p_r}{p_s} \equiv \overline{\mathcal{N}}_1 \ .$$

Previous assumptions ($\overline{R}_s > \overline{R}_r$) only guarantee $\mathcal{N}$ is strictly positive, so for $\mathcal{N} \in (0, \overline{\mathcal{N}}_1)$, individual loans give rise to inefficiency, namely risky project choice or none at all.

Given group contract $(r_0, r_1, ..., r_{n-1})$, the payoff of a borrower all of whose group chooses the safe project is Π_s (equation 3). If all groups choose the safe project, the ZPC is

$$\sum_{k=0}^{n-1} p_s^{n-k}(1-p_s)^k \binom{n-1}{k} r_k \geq \rho \ ;$$

at equality, this pins down the safe payoff at $\overline{R}_s - \rho$, a level higher than the outside option.

[22] $\mathcal{N}$ and $\mathcal{G}$ are redefined in this section only, to make clear the tight connection between the results here and those under adverse selection.

A key incentive compatibility (IC) constraint must be satisfied:

$$\overline{R}_s - \sum_{k=0}^{n-1} p_s^{n-k}(1-p_s)^k \binom{n-1}{k} r_k \geq \overline{R}_r - \sum_{k=0}^{n-1} p_r^{n-k}(1-p_r)^k \binom{n-1}{k} r_k \ . \tag{6}$$

This IC constraint rules out the entire group collectively switching to risky projects. It also guarantees that switching a strict subset of members to the risky project does not raise the group payoff under an affordable full liability contract:

Lemma 5. *Under an affordable full liability contract with $r \in (0, R_s)$, IC constraint 6 guarantees that a group cannot achieve a higher sum of payoffs by undertaking any number of risky projects, the remainder being safe, than it can by choosing all safe projects.*

This lemma is the analog to the homogeneous-matching result in the adverse selection case, and it relies on similar properties of the payoff function – supermodularity and symmetry.

Note also the critical assumption in IC constraint 6 that the group acts cooperatively, to maximize total group payoffs. Thus the constraint rules out a coordinated shift toward risky projects, but not a unilateral deviation.[23] This is the same assumption made and argued for by Stiglitz (1990), without which group lending in this context offers no improvement (see also Conning, 2005). Thus, enforceable, cooperative behavior in project choice is the analog here to local information on risk-type in the adverse selection setting. Without it, group size is irrelevant here as it is under random matching in the adverse selection setting – implying that, again, simple Central Limit Theorem insurance-based logic is not sufficient.

We next show a result similar to Proposition 2 that only requires $\mathcal{G}$ – defined in this section only as $\overline{R}_s/\rho$ – to exceed a fixed bound independent of n.

Proposition 3. *A large enough group size n and assumption A3 guarantee maximal borrower surplus and complete outreach are achieved by an affordable full liability group contract.*

[23]Though written in terms of individual payoffs, the constraint is equivalent to a group-level constraint, that the sum of payoffs when all choose safe is no less than the sum of payoffs when all choose risky. We also have shown that imposing this constraint rules out any improvement to the *group payoff* from one member (or more) switching to risky project choice under an affordable full liability contract. But, it will not rule out a unilateral deviation where the deviator enjoys higher payoffs than the rest of the group by essentially free-riding on their safe behavior.

The intuition is familiar. Larger groups and affordable full liability move toward certainty the likelihood that a group will pay its entire obligation – this allows for asymptotic elimination of the implicit subsidy to groups that choose risky projects under limited liability. Again, affordability of the entire group obligation need not obtain in all cases, but only under mean failure rates, since these events dominate as n gets large.

But again, large groups are not enough. Cooperative project choice is also needed to ensure the group is all making the same choice (as is collectively optimal) – this eliminates within-group subsidies to risk-taking. If instead unilateral deviations were allowed, then a borrower deviating to a risky project would enjoy a within-group subsidy to risk, and this temptation would undermine the gains from group lending.

6.2 *Ex post* Moral Hazard and Group Size

Here we use the framework of Baland et al. (2013; "BSW"). Each borrower is endowed with an identical project that requires one unit of capital and pays R when it succeeds – with probability $p \in (0, 1)$ – and pays 0 otherwise; the expected payoff is $\overline{R}$. Contracts must satisfy limited liability and lender break-even at gross rate $\rho > 0$.[24]

A key difference from our model is that output is unobserved by the lender, implying repayment cannot be directly enforced. The lender gives repayment incentives by imposing a non-pecuniary penalty K on a borrower who defaults. This implies that a successful borrower will repay as long as K exceeds the amount due, and otherwise default. Lender enforcement capacity is limited, i.e. K is capped by $\overline{K} > 0$. Penalties are necessary to support lending, but they lead to inefficiency since they are imposed on unlucky borrowers in equilibrium. Thus, borrower welfare is tightly connected to minimizing equilibrium imposition of penalties while making repayment incentive compatible.

Heterogeneity in borrower wealth, $w \in [0, 1)$, the second key difference from our model, implies that borrowers have different needs for outside capital to fund their projects. Rela-

[24]The outside labor option is zero.

tively wealthy agents can self-finance most of the project, while relatively poor agents will need larger loans. However, given limited enforcement capacity of the lender, there is a limit on feasible loan size. Thus, relatively poor agents may be unable to borrow.

Consider individual lending. A contract with loan size L and no strategic default involves break-even interest rate $r = \rho/p$ and requires $K \geq L\rho/p$ to prevent strategic default. This implies that the maximum supportable loan size is $p\overline{K}/\rho \equiv \overline{L}$. Note that $\overline{L} < 1$ under BSW's assumption

$$\overline{K} < \frac{\rho}{p} .$$

Thus, given limited enforcement, individual loans cannot reach low-wealth borrowers (those with $w < 1 - \overline{L}$).

Maximal payoffs for those who can borrow are

$$\Pi_I(w) = \overline{R} - \rho w - pL\rho/p - (1-p)K = \overline{R} - \rho - \frac{1-p}{p}\rho(1-w) .$$

The first expression subtracts opportunity cost of own capital, loan payment with probability p, and penalty K with probability $1-p$; the second substitutes in the minimum penalty given loan size L, i.e. $L\rho/p$, and the minimum loan size $L = 1 - w$. BSW's assumption that

$$\overline{R} \geq \rho + (1-p)\overline{K}$$

guarantees that all borrowers with $w \geq 1 - \overline{L}$ get positive payoffs from borrowing. Still, payoffs are suboptimal; they equal the first-best payoff, $\Pi_{FB} = \overline{R} - \rho$, minus expected penalties – interest rate ρ/p, times loan size $1-w$, times probability of failure $1-p$.

The focus of BSW is on how full liability group loans compare to individual loans in both outreach – i.e. maximum feasible loan $\overline{L}$ – and borrower welfare – i.e. payoffs for those who do borrow. In addition, they consider how the availability of social sanctions and the size of the group affect these comparisons. With respect to group size, however, they focus mainly on non-monotonicities rather than exploring outcomes when group size can be set freely. Our goal here is to supplement their work, and in so doing to show robustness of the logic of this paper in a different setting.

The key result of BSW assuming no borrower social capital (BSW-Proposition 2) is that group lending achieves lower outreach than individual lending, but can sometimes provide higher welfare for the borrowers it does reach (the wealthier ones). We show here that when group size is unrestricted, group lending can achieve arbitrarily close to the level of outreach that individual loans can, and can eliminate arbitrarily much of the gap between individual loan payoffs and first-best payoffs. Thus, group lending can provide better payoffs for almost every agent. To show this, let $\Pi_G(w, n)$ be borrowing payoffs for an agent with wealth w in a borrowing group of size n, let $GAP_G(w, n) \equiv \Pi_{FB} - \Pi_G(w, n)$ be the shortfall from first-best payoffs in such a group contract, and let $GAP_I(w) \equiv \Pi_{FB} - \Pi_I(w) = \rho(1-w)(1-p)/p$ be the shortfall from first-best payoffs in an individual contract.

Proposition 4. *Fix any $\overline{L}' \in (0, \overline{L})$ and $\epsilon \in (0,1)$. A large enough group size n guarantees that a group contract can feasibly lend all loan sizes $L \in (0, \overline{L}')$ and, for all borrowers, eliminate all but ϵ of the inefficiency in individual loan payoffs ($GAP_G(w, n) < \epsilon\, GAP_I(w)$).*

Thus group lending can be made to dominate individual lending for all but an arbitrarily small interval of borrower wealth levels, by choosing an adequately large group size. The intuition is that larger groups cause the the probability of group default to vanish, and with it the equilibrium imposition of penalties (raising borrower payoffs) and the chance that a successful borrower chooses to default (raising outreach).

This result may seem to contradict BSW-Proposition 6, which shows that welfare can sometimes be increased by decreasing group size. But BSW-Proposition 6 is a local result establishing the possibility of non-monotonicity; it says that in some cases, borrower welfare can be higher under group size $n' < n$ than under group size n. What Proposition 4 implies is that there also exists a group size $n'' > n$ such that group size n'' gives borrowers higher welfare than group sizes n and n'. In other words, while in some cases welfare can be improved with a smaller group, it can be improved *even more* by a larger group.[25]

BSW also introduce social sanctions, as an extra penalty γ that can be imposed on a group member who defaults even though he can repay. This extra penalty raises a borrower's willingness to repay because his total sanction after strategic default is $K + \gamma$. It is also an attractive penalty because it is never imposed in equilibrium – unlike the lender, borrowers can observe each other's output realizations and do not impose sanctions after project failure.

BSW find that the availability of social sanctions can improve outreach and borrower welfare under group lending; more surprisingly, they find that however strong the social sanctions, group lending may not achieve greater outreach than individual loans if p is high enough (BSW-Proposition 4). This seemingly places a limit on how well group lending can do in some cases, even with arbitrarily large social sanctions available.

But this is with fixed group size. We show here that for any p and *any* amount of social sanctions, group lending with adequately large groups strictly dominates individual lending in both outreach and payoffs, achieving borrower payoffs arbitrarily close to first-best. Further, if social sanctions are large enough, group contracts can get arbitrarily close to fully efficient lending – complete outreach and borrower payoffs arbitrarily close to first-best.

Proposition 5. *A) Fix any $\epsilon \in (0,1)$ and any social sanction $\gamma > 0$. There exists an $\overline{L}' > \overline{L}$, such that a large enough group size n guarantees that a group contract can feasibly lend all loan sizes $L \in (0, \overline{L}')$ and, for all borrowers, eliminate all but ϵ of the inefficiency in individual loan payoffs ($GAP_G(w,n) < \epsilon\, GAP_I(w)$).*

B) Fix any $\epsilon \in (0,1)$ and assume $\gamma > \rho/p - \overline{K}$. A large enough group size n guarantees that a group contract can feasibly lend all loan sizes $L \in (0,1]$ and, for all borrowers, eliminate all but ϵ of the inefficiency in individual loan payoffs ($GAP_G(w,n) < \epsilon\, GAP_I(w)$).

The intuition for A is clear, since even without social sanctions, group lending can get arbitrarily close to individual lending in outreach; a little extra social enforcement is enough to give group lending the advantage. The intuition for B is that if the social sanction plus

[25]This is also seems to be illustrated in the BSW simulations; see their Figure 1.

bank sanction are enough to incentivize repayment of ρ/p, $(\overline{K}+\gamma > \rho/p)$, then large groups can do the rest, ensuring that with near certainty this is the amount owed.[26]

In short, we find that large enough group size gives group lending a significant advantage over individual lending even without any social capital; and with sufficient social capital, group lending approaches fully efficient lending. These results are remarkably similar to those obtained in the other contexts. Perhaps the most striking difference is that in this context as opposed to the other two, group lending with large enough groups does not require *any* social capital to significantly improve borrower welfare compared to individual lending.[27] It is also striking that in all three contexts, group lending with adequately-sized groups, in conjunction with a sufficient social asset, approaches fully efficient lending.

7 Realistic Group Sizes

The results of this paper show that affordable full liability contracts with large enough groups achieve efficient lending, or arbitrarily close to it. But, how large is large enough? This is a relevant question because the larger is group size, the less plausible are some of the papers' key assumptions. Namely, the quality of local information and/or the ability to enforce cooperative agreements and impose social sanctions may deterioriate as group size gets large. If so, efficiency may not always be attainable, and tradeoffs may exist that give an interior optimal group size.

The goal of this section is two-fold. First, we formalize the preceding argument theoretically in the baseline context of adverse selection. We assume information quality (in particular, the ability to match homogeneously) deteriorates with group size, and show that an interior group size does better than either extreme.[28] Second, we simulate the model and show that the benefits of larger group size obtain for reasonable group sizes. Thus, the logic

[26] Note that if the interest rate is ρ, then in a "large" group a successful borrower owes $\rho + \rho(1-p)/p$, the latter term capturing his share of bailout payments. This equals ρ/p.

[27] However, it may be that an assumption on repayment coordination is implicit.

[28] Conning (2005) also shows an interior optimal group size in a moral hazard context where social sanctions available to enforce cooperative agreements are bounded.

of the asymptotic results is applicable not only in the limit.

7.1 Information Quality and Group Size

While a full analysis is beyond the scope of this paper, we explore briefly the question of how optimal group size is determined when information quality declines with group size. Information quality is formalized in a somewhat mechanical way; the thrust of the assumptions, though, is simply that larger group size makes homogeneous matching more difficult.[29]

We imagine that a matching equilibrium occurs, along with irreversible borrowing decisions; then, before payoffs are realized, each borrower has an identical and independent probability of being removed from his group and being re-placed in another randomly selected borrowing group. The replacement process preserves group size. Every borrower's probability of *not* being replaced is $\pi(n)$, with $\pi(n) > 0$ and $\lim_{n\to\infty} \pi(n) = 0$. Thus, the expected fraction of type-p_i borrowers that remain after replacement in an initially homogeneous type-p_i group is strictly positive, but approaches zero in n. Borrowers anticipate the replacement process in the group formation process and in calculating payoffs to decide whether to borrow.

We again restrict attention to *ex ante* (i.e. pre-replacement) homogeneous group formation, and since our focus will be on when fully efficient lending is achievable, we write the conditions corresponding to all types borrowing. The *ex post* (i.e. post-replacement) expected type of a fellow group member of a type-p_i borrower is

$$\tilde{p}_i \equiv \pi(n)\, p_i + [1 - \pi(n)]\overline{p} \ ,$$

where $\overline{p}$ is the average risk-type and dependence of $\tilde{p}_i$ on n is suppressed. This expression holds since with probability $1 - \pi(n)$ the initially identical borrower has been replaced with

[29] This could be the simple consequence of living in finite-sized communities with only a minority of households interested in borrowing. The larger the group size, the less homogeneous groups would be even with frictionless matching. Introducing search/matching frictions could amplify this effect.

a borrower drawn at random from the pool of borrowers, i.e. from the entire distribution of types. Since replacement happens independently across all borrowers, one can essentially consider a fellow group member's ex post type to be $\tilde{p}_i$ if his ex ante type was p_i.

Consider the ex post payoff of a borrower of type i. If he is not replaced, it is

$$\overline{R} - p_i \sum_{k=0}^{n-1} \tilde{p}_i^{n-k-1}(1-\tilde{p}_i)^k \binom{n-1}{k} r_k \ ,$$

since he pays r_k iff he succeeds – probability p_i – and k of his $n-1$ fellow group members fail – probability $\tilde{p}_i^{n-1-k}(1-\tilde{p}_i)^k\binom{n-1}{k}$. If he is replaced into a group of ex ante type j, his payoff is the same as the above with $\tilde{p}_j$ in place of $\tilde{p}_i$; since he is equally likely to be replaced into any group, his payoff if replaced involves the expectation over all types $\tilde{p}_j$:

$$\overline{R} - p_i \sum_{k=0}^{n-1} \overline{\tilde{p}^{n-k-1}(1-\tilde{p})^k} \binom{n-1}{k} r_k \ .$$

Since the replacement probability is $1-\pi(n)$, the expected payoff is

$$\overline{R} - p_i \sum_{k=0}^{n-1} \left[\pi(n)\, \tilde{p}_i^{n-k-1}(1-\tilde{p}_i)^k + (1-\pi(n))\, \overline{\tilde{p}^{n-k-1}(1-\tilde{p})^k}\right] \binom{n-1}{k} r_k \ . \tag{7}$$

The zero-profit constraint can be derived analogously:

$$\begin{aligned} \sum_{k=0}^{n-1} \left[\pi(n)\, \overline{p\tilde{p}^{n-k-1}(1-\tilde{p})^k} + (1-\pi(n))\, \overline{p}\overline{\tilde{p}^{n-k-1}(1-\tilde{p})^k}\right] \binom{n-1}{k} r_k = \\ \sum_{k=0}^{n-1} \left[\overline{\tilde{p}^{n-k}(1-\tilde{p})^k}\right] \binom{n-1}{k} r_k \geq \rho \ . \end{aligned} \tag{8}$$

Proposition 6. *If $\mathcal{G} > p_s/\overline{p}$ and if local information quality is positive but approaching zero as groups get large, then there exist integers n^* and $\overline{n}$, with $1 < n^* < \overline{n}$, such that fully efficient lending is achieved over a strictly larger parameter space $(\overline{R}, \overline{u}, \rho)$ by group lending with $n = n^*$ than by both a) individual lending and b) group lending with $n \geq \overline{n}$.*

Thus, the best shot at full efficiency is by group lending with a finite group size.[30]

This result formalizes a potentially countervailing negative aspect of larger groups, the deterioration of match quality. If matching degenerates toward random matching as groups get large enough (i.e. $\lim_{n\to\infty} \pi(n) = 0$), then large groups lose their value, approaching equivalence with individual lending (see Lemma 4). This guarantees that some interior group size with a somewhat effective matching outcome will outperform both individual lending and group lending with large but ineffective groups.

The rate at which matching deteriorates – more generally, the shape of $\pi(n)$ – will no doubt play a significant role in optimal group size. Thus, one might expect smaller optimal group sizes when, all else equal, the population within which matching occurs is smaller, and also when information is relatively localized or limited within a given matching population.

7.2 Simulations

We turn to numerical simulations of the model to assess how well the theoretical mechanism works at reasonable group sizes. The model has a limited number of parameters, which can be broken into two categories, risk and return, respectively. The risk parameters are the upper and lower bound on risk-type, p_r and p_s, and the distribution function $f(p)$. The return parameters are the expected gross project return, $\overline{R}$, the return to labor without borrowing, $\overline{u}$, and the lender's required return to capital ρ. Equivalently, the return parameters can be summarized with ρ, gross excess return $\mathcal{G}$ $(\overline{R}/\rho)$, and net excess return $\mathcal{N}$ $((\overline{R}-\overline{u})/\rho)$.

The literature gives a fair amount of guidance on reasonable return parameters. For ρ, we assume 1.1, meaning the microlender requires an expected 10% return to break even. This follows de Quidt et al. (2012a, 2012b), and appears reasonable given the higher operating costs required for microlending (see e.g. Ahlin et al., 2011). Recent literature estimating returns to capital in developing countries helps shed light on the project return parameters,

[30] At the expense of significantly more notation, it is also possible to show a related result, that borrower surplus is maximized by an interior group size.

where MPK estimates for microentrepreneurs in the 60% to 100% range are not uncommon.[31] A complication arises in how to interpret these estimated returns to capital – are they estimated net of the full opportunity cost of labor, or not? Often, it is difficult empirically to fully account for the increased or subsituted labor/effort required to use the additional capital. The question for this study is whether the rates of return in the literature apply to the gross or net excess return, $\mathcal{G}$ or $\mathcal{N}$. The issue is not too critical, however, since rates of return appear to vary a lot across developing country contexts, even within countries.[32] Accordingly, we allow for various levels of both $\mathcal{N}$ and $\mathcal{G}$ in the simulations. Still, in line with our reading of the empirical work, we concentrate on $\mathcal{G}$ in the upper ranges here (1.7 to 2.3) and focus on $\mathcal{N}$ below these ranges.

The literature is much less developed on risk parameters, especially since here the key parameters are on heterogeneity in unobserved project risk across households. Our approach here is to assume a standard distribution – the uniform – and what appear to be reasonable bounds on risk-types: $[0.5, 0.99]$. We verify the reasonability of the default rates that this distribution delivers.

FIGURE 1 ABOUT HERE (see Appendix)

Our main result was that for any $\mathcal{N} > 1$ and $\mathcal{G} > p_s/p_r$, fully efficient lending – equivalently, complete outreach – is attained by large enough groups. The left panel of Figure 1 shows how the cutoff $\mathcal{N}$, above which complete outreach is achieved, varies with n. It does so for $\mathcal{G} \in \{1.7, 2.0, 2.3\}$, which essentially varies affordability of full liability. One can see that in all cases, larger groups extend the parameter space over which complete outreach is possible. It seems clear that the $\mathcal{N}$ cutoff is approaching 1 for $\mathcal{G} \in \{2.0, 2.3\}$, both of which values satisfy $\mathcal{G} > p_s/p_r (= 1.98)$. However, when $\mathcal{G} = 1.7$, which does not satisfy $\mathcal{G} > p_s/p_r$, the cutoff is not approaching 1 – there will remain a neighborhood of $\mathcal{N}$ above 1 such that fully efficient lending is not achieved no matter how large are groups.

[31] De Quidt et al. (2012a, 2012b) use a 60% rate of return, while Banerjee and Duflo (2005) use 100% for small firms.

[32] This a major theme of Banerjee and Duflo (2005).

Default rates are also reasonable above the $\mathcal{N}$ cutoffs, at least as group size moves away from 1 and 2, equaling about 10% when the cutoff is 1.1 and 5% when it reaches 1.05, and approaching 0% as the cutoff approaches 1. At group size of 5, default above the $\mathcal{N}$ cutoff equals 9%, 6%, and 4%, respectively, for $\mathcal{G} \in \{1.7, 2.0, 2.3\}$. What is also clear in Figure 1 is that much of the action is over the range of reasonable group sizes – only for net excess returns below 2-3% can a group of size 5-10 not achieve full outreach, and this even if Assumption A3 barely holds ($\mathcal{G} = 2.0$). Further, the simulations show that for a fixed $\mathcal{N}$, greater affordability (higher $\mathcal{G}$) can substitute for larger group size, lowering the need for large groups in order to achieve fully efficient lending.[33]

The right panel of Figure 1 is for the model of Section 7.1, where local information deteriorates with group size. Specifically, we let the probability of remaining in one's group be $\pi(n) = e^{\lambda(1-n)}$, with $\lambda = 0.02$. This implies that in a group of size 10, on average just under 2 members are of the wrong risk-type. Figure 1 demonstrates that, as anticipated from Proposition 6, the group size that achieves complete outreach over the greatest parameter space is finite. Interestingly, it is also quite moderate – 8, 6, and 4, respectively, for $\mathcal{G} \in \{1.7, 2.0, 2.3\}$. Default rates are similar to the first panel, showing that large groups are still eliminating group default – but as was shown, the problem is that the within-group cross-subsidy is growing as matching deteriorates, working against efficiency.

The first set of simulations finds conditions for fully efficient lending – complete outreach – as group size varies. The second set solves for maximum achievable outreach as a function of group size, for given $\mathcal{N}$. Figure 2 graphs outreach – percent of total borrowers that are reached at the optimum – against group size, using the same three values of $\mathcal{G}$ and two panels, with and without information frictions. $\mathcal{N}$ is set to 1.1, i.e. net excess return is 10%.

FIGURE 2 ABOUT HERE (see Appendix)

In the left panel, we see a similar theme, that large groups allow for greater outreach (efficiency), and that reasonable group sizes are able to attain full efficiency. Most surprising

[33]De Quidt et al. (2012a, b) comment on a similar phenomenon in their ex post moral hazard context.

is how dramatically outreach can increase once a critical group size is reached. Outreach jumps from relatively low levels to 100% once a critical group size is reached – 5, 4, and 2, respectively, for $\mathcal{G} \in \{1.7, 2.0, 2.3\}$. This jump is occurring here because it is easiest to attract either the left tail or the entire distribution. As group size increases, risk-pricing improves and the size of the left tail that can be attracted gradually increases; but at the same time, the entire distribution is getting closer to within reach. When it is finally attainable, outreach jumps to 100%.

The results with information frictions, in the right panel, are similar in that past a cutoff group size, outreach skyrockets to 100%. The critical group sizes are 4 for $\mathcal{G} = 2.3$ and 5 for $\mathcal{G} = 2.0$. Of course, the results differ in that outreach begins to decline as group size continues to grow – beyond 8 or 9 for $\mathcal{G} = 2.0$ and $\mathcal{G} = 2.3$, respectively. However, the decline is more gradual than the buildup; it appears more costly to be a bit smaller than optimal than a bit larger.

In sum, we draw several conclusions from the simulations. First, the asymptotic arguments are applicable at reasonable group sizes, and even in some cases when the affordability assumption is not met. Second, higher gross returns relax affordability and can substitute for larger group size. Third, there can be a "discontinuity" in efficiency as group size increases, with efficiency jumping substantially as a typically moderate threshold is reached. Overall, the results help rationalize observed group sizes, which are often four to ten, as not just incidental but perhaps critical to the effective use of group lending.

8 Conclusion

We have explored the usefulness of what is perhaps an underemphasized tool that lenders have in improving lending efficiency – group size. Our main progress has been in reaching a deeper understanding of the benefits of larger group size, not because we believe the costs to be unimportant, but because as a first step we have mostly taken the models and assumptions off the shelf. However, we do explore costs of larger groups as well, sketching an approach toward modeling optimal group size. More work bringing costs and benefits together is certainly called for.

The results may be taken not as a critique of the majority of the literature with group size $n = 2$, but as a reminder that this assumption shuts down one potentially important tool the lender has to increase lending efficiency. Further, since group size is shown here to be a tool that can interact in interesting ways with other tools the lender may be able to harness (e.g. social capital), there is also a caveat here that statements made for smaller groups may not hold at larger group size.

Beyond the scope of this paper are two factors that may be important in relevant lending environments. First, borrowers are assumed risk-neutral, not risk-averse. Intuitively, large groups would have an additional appeal under risk aversion, since they help stabilize borrower payoffs via intragroup insurance. Thus large groups can also be a way to decrease the added risk associated with group lending (a drawback pointed out by Stiglitz, 1990). Second, projects are assumed to be statistically independent here. We conjecture, however, that correlated risk that affects all types symmetrically need not change the results. Even if it meant infinitely large groups could default with positive probability due to correlated risk, as long as this probability is the same across safe and risky groups, there would be no resulting cross-subsidy. Hence, the extra default risk could be priced into all borrowers' loans uniformly, and risk-pricing would still be asymptotically perfect. But, if correlated risk changed group default probabilities unevenly across safe and risky borrowers, and if it were great enough to imply that groups default with a probability bounded away from zero, then it seems perfectly efficient lending would no longer be achievable. Future work can take up these two issues more fully.

The paper suggests a number of directions for empirical work. The implications about complementarity between social capital and group size in improving lending outcomes could be explored; among other goals, they might be useful in testing across models, since the degree of complementarity seems to vary with the underlying frictions. The results also can be used to form hypotheses about observables in an environment, e.g. village population or density, social information and enforcement capabilities, or rates of return, that help determine the costs and benefits of larger group size along lines suggested by this theory.

In short, the paper argues that larger groups can take greater advantage of local information in an adverse selection setting, and local coordination/enforcement in a moral hazard setting – though there are likely to be limits. From a positive standpoint, this helps explain why observed group sizes are almost invariably larger than two. It also points to potential factors behind optimal group size that can give rise to some of the variation observed in contemporary micro-lending.

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Appendix

Notation, and Central Limit Theorem (CLT) Fact and Corollary.

For any $n > 0$, let $G \in [p_r, p_s]^n$ be n agent types. Let $B_{n,j,G}$ denote the probability of j or more successes in n independent trials, where the respective success probabilities of the n trials come from the n types in G. Let $B_{n,j,p}$ be the probability of j or more successes in n independent trials all with success probability p. Similarly, let $\tilde{B}_{n,j,G}$ and $\tilde{B}_{n,j,p}$ be the probabilities of *exactly* j successes in n independent trials with the success probabilities as defined in $B_{n,j,G}$ and $B_{n,j,p}$, respectively.

Consider an affordable full liability contract ("AFLC"; equation 5) with group size n and interest rate $r \in [0, R_s]$, and define $k^*(n,r)$ as the minimum number of group failures for which full liability is unaffordable (or just affordable); more precisely, partial brackets $\lceil\ \rceil$ denoting the ceiling function, let

$$k^*(n,r) \equiv \left\lceil n\left(1 - \frac{r}{R_s}\right)\right\rceil \ . \tag{9}$$

The following **CLT Fact** follows from the CLT (e.g. DeGroot, 1986, p.275). The proof is standard and available upon request. For some function $j(n)$,

$$\begin{aligned} \lim_{n\to\infty} \frac{j(n)}{n} < p &\implies \lim_{n\to\infty} B_{n,j(n),p} = 1 \ . \\ \lim_{n\to\infty} \frac{j(n)}{n} > p &\implies \lim_{n\to\infty} B_{n,j(n),p} = 0 \ . \end{aligned} \tag{10}$$

CLT Corollary. Assume $p_s R_s/\rho > p_s/p_r$; equivalently, $\mathcal{G} > p_s/p_r$. For any $p \in [p_r, p_s]$,

$$\begin{aligned} \lim_{n\to\infty} r(n) = \rho &\implies \lim_{n\to\infty} B_{n,n-k^*(n,r(n))+1,p} = 1 \ . \\ \kappa \in (1, \mathcal{G}/(p_s/p_r)) &\implies \lim_{n\to\infty} B_{n,n-k^*(n,\kappa\rho)+1,p} = 1 \ . \end{aligned}$$

Proof. Since $p_r \le p$, by the CLT Fact it is sufficient to show that, assuming both hypotheses,

$$\max\left\{\lim_{n\to\infty} \frac{n - k^*(n,r(n)) + 1}{n}, \lim_{n\to\infty} \frac{n - k^*(n,\kappa\rho) + 1}{n}\right\} < p_r \ , \qquad i.e., \qquad \frac{\kappa\rho}{R_s} < p_r \ ,$$

which is guaranteed by $\kappa < \mathcal{G}/(p_s/p_r)$. ■

Proof of Lemma 1. Monotonicity guarantees that group payments are increasing in the number of successes. The distribution of number of successes of safer groups stochastically dominates that of riskier groups. Thus, in expectation safer *groups* pay more and earn lower net payoffs. Since agent payoffs in homogeneous-matching equilibrium are $1/n$ times group payoffs, in expectation safer *borrowers* pay more and earn lower net payoffs. Thus, if an agent of type $\hat{p}$ borrows, he earns more than $\overline{u}$; so do all riskier types, and thus they also borrow. ■

Proof of Lemma 2. The Proof uses notation defined in the beginning of the Appendix. The Proof consists of demonstrating strictly positive cross-partials of the group payoff function, Π_G; since Π_G is smooth in the group risk-type p_i's, homogeneous group formation is then guaranteed as the unique stable match (Ahlin, 2012, Proposition 2).

First consider the case where $n = 2$, and group types are (p_x, p_y). One can write the group payoff function and cross-partial, respectively, as

$$\Pi_G = 2\overline{R} - (p_x + p_y)r_1 + 2p_x p_y(r_1 - r_0) \qquad \textit{and} \qquad \frac{\partial^2 \Pi_G}{\partial p_x \partial p_y} = 2(r_1 - r_0) \ .$$

This is strictly positive since $r \in (0, R_s)$, which guarantees $r_0 < r_1$ in an AFLC.

For the remainder of the proof, fix $n > 2$ and set of n types $G = (p_1, p_2, ..., p_n) \in [p_r, p_s]^n$. Let N be $\{1, 2, ..., n\}$. Suppressing dependence of k^* on r and n, the group payoff function from borrowing under an AFLC for this set of n types is

$$\Pi_G = n\overline{R} - nrB_{n,n-k^*+1,G} - \sum_{j=k^*}^{n}(n-j)R_s\tilde{B}_{n,n-j,G} \ ,$$

since the lender gets nr if there are less than k^* failures (more than $n - k^*$ successes), and otherwise, if there are j failures, gets R_s from $n - j$ borrowers.

First consider the case where $k^* = n$. The group payoff can then be written

$$\Pi_G = n\overline{R} - nrB_{n,1,G} = n\overline{R} - nr\left[1 - \prod_{j \in N}(1 - p_j)\right] \ ,$$

since the group pays nr unless all members fail. Choosing any $x, y \in N$, the cross-partial is

$$\frac{\partial^2 \Pi_G}{\partial p_x \partial p_y} = nr \prod_{i \in N \setminus \{x,y\}} (1 - p_i) > 0 \ .$$

Thus all cross-partials are strictly positive.

Finally, consider the case in which $k^* \leq n - 1$, and note that $r < R_s$ implies $k^* \geq 1$. Choose any $x, y \in N$ and let $\tilde{G}$ be a vector of $n - 2$ types, equaling G with one borrower each of type p_x and p_y removed. Note that

$$B_{n,n-k^*+1,G} = B_{n-2,n-k^*+1,\tilde{G}} + (p_x + p_y - p_x p_y)\tilde{B}_{n-2,n-k^*,\tilde{G}} + p_x p_y \tilde{B}_{n-2,n-k^*-1,\tilde{G}} \ .$$

because the number of group successes is at least $n - k^* + 1$ iff one of three disjoint events occurs: the number of successes in the rest of the group $\tilde{G}$ is at least $n - k^* + 1$; the number of successes in $\tilde{G}$ is $n - k^*$ and at least one of borrowers x and y succeed; and the number of successes in $\tilde{G}$ is $n - k^* - 1$ and both borrowers x and y succeed. From this, it is clear that

$$\frac{\partial^2 B_{n,n-k^*+1,G}}{\partial p_x \partial p_y} = \tilde{B}_{n-2,n-k^*-1,\tilde{G}} - \tilde{B}_{n-2,n-k^*,\tilde{G}} \ .$$

By similar reasoning,

$$\tilde{B}_{n,n-j,G} = p_x p_y \tilde{B}_{n-2,n-j-2,\tilde{G}} \ + \ [p_x(1-p_y) + p_y(1-p_x)]\tilde{B}_{n-2,n-j-1,\tilde{G}} \\ + \ (1-p_x)(1-p_y)\tilde{B}_{n-2,n-j,\tilde{G}} \ ,$$

so

$$\frac{\partial^2 \tilde{B}_{n,n-j,G}}{\partial p_x \partial p_y} = \tilde{B}_{n-2,n-j-2,\tilde{G}} - 2\tilde{B}_{n-2,n-j-1,\tilde{G}} + \tilde{B}_{n-2,n-j,\tilde{G}} \ .$$

One can use this to show the following simplification:[34]

$$\sum_{j=k^*}^{n} \frac{\partial^2 \tilde{B}_{n,n-j,G}}{\partial p_x \partial p_y}(n-j) = (n-k^*)\tilde{B}_{n-2,n-k^*,\tilde{G}} - (n-k^*+1)\tilde{B}_{n-2,n-k^*-1,\tilde{G}} \ .$$

Putting these facts together gives

$$\frac{\partial^2 \Pi_G}{\partial p_x \partial p_y} = [R_s(n-k^*+1) - nr]\tilde{B}_{n-2,n-k^*-1,\tilde{G}} + [nr - R_s(n-k^*)]\tilde{B}_{n-2,n-k^*,\tilde{G}} \ .$$

Using the definition of k^* from equation 9, there exists an $\epsilon \in [0,1)$ such that

$$k^* = n\left(1 - \frac{r}{R_s}\right) + \epsilon.$$

Incorporating this into the previous expression gives

$$\frac{\partial^2 \Pi_G}{\partial p_x \partial p_y} = R_s\left[(1-\epsilon)\tilde{B}_{n-2,n-k^*-1,\tilde{G}} + \epsilon\tilde{B}_{n-2,n-k^*,\tilde{G}}\right] > 0 \ ,$$

the strict inequality because $(1-\epsilon) > 0$ and $1 \leq k^* \leq n-1$, so $\tilde{B}_{n-2,n-k^*-1,\tilde{G}} > 0$. Thus all cross-partials are strictly positive. ■

Proof of Proposition 1. Define an "admissible group contract" as a group contract satisfying monotonicity, homogeneous matching, and limited liability. Partly due to Lemma 2, all AFLCs with $r \in (0, R_s)$ are admissible group contracts.

We will show that for any admissible group contract C satisfying the ZPC, there is an admissible AFLC satisfying the ZPC and delivering at least as much borrower surplus. Fix such a contract C. First, consider the case that C attracts no borrowers. To be admissible C must involve $r_k < R_s$ for at least one $k \in \{0, 1, ..., n-1\}$ (otherwise unique homogeneous matching does not obtain). Then, there is clearly an AFLC with r close enough to R_s that give all borrowers worse payoffs, and thus under which no borrowers opt in.

Second, consider the case where C attains maximal borrower surplus. This implies it satisfies ZPC 4. Now, one can show that the admissible group contract that maximizes

[34] This simplifies in part because terms $\tilde{B}_{n-2,m,\tilde{G}}$ where $1 \leq m \leq n-k^*-2$, if such exist, appear three times in the sum (when $j = m, m+1, m+2$) with coefficients that cancel out; $\tilde{B}_{n-2,0,\tilde{G}}$ appears twice (unless $k^* = n-1$) with coefficients that cancel out; and $\tilde{B}_{n-2,-1,\tilde{G}} = 0$.

the safest ("safe") borrower's payoff subject to ZPC 4 is the AFLC that satisfies ZPC 4 at equality.[35] To see this, note that the ZPC slope in (r_{k-1}, r_k) space is

$$\frac{dr_k}{dr_{k-1}} = -\frac{\overline{p^{n-k+1}(1-p)^{k-1}}\binom{n-1}{k-1}}{\overline{p^{n-k}(1-p)^{k}}\binom{n-1}{k}}$$

and the slope of the safe-borrower indifference curve is, using payoff expression 3,

$$\frac{dr_k}{dr_{k-1}} = -\frac{p_s^{n-k+1}(1-p_s)^{k-1}\binom{n-1}{k-1}}{p_s^{n-k}(1-p_s)^{k}\binom{n-1}{k}} = -\frac{p_s\binom{n-1}{k-1}}{(1-p_s)\binom{n-1}{k}}.$$

Both are linear, and one can show that the indifference curve is steeper than the ZPC.[36] Thus, ignoring constraints, the safe-borrower payoff can be strictly raised by increasing r_k and lowering r_{k-1} along the ZPC. This implies that the admissible group contract that maximizes the safe-borrower payoff subject to the ZPC sets all interest rates r_k, $k \in \{1, 2, ..., n-1\}$, at their upper bounds, which are provided by the monotonicity or limited liability constraints. (If not, there would exist an r_k strictly below its upper bounds and a corresponding r_{k-1} strictly above its lower bound, which could be adjusted along the ZPC to raise safe-borrower payoffs.) Equivalently, it is an AFLC. Further, it is the one that satisfies ZPC 4 at equality; otherwise r could be lowered while satisfying all constraints, raising safe payoffs.

Again, since C attains maximal borrower surplus, it satisfies ZPC 4 and delivers the safe borrower a borrowing payoff of at least $\overline{u}$. Now consider the AFLC that satisfies ZPC 4 at equality. It also delivers the safe borrower a payoff of at least $\overline{u}$, since it is the admissible contract that maximizes the safe borrower payoff subject to ZPC 4. By Lemma 1, all other types also borrow under this AFLC. Hence, all borrowers are included and maximal borrower surplus is attained by the AFLC satisfying ZPC 4 at equality.

Finally, consider the case where C attracts some but not all agents. By Lemma 1, the set of borrowing types can be written $[p_r, \hat{p}]$, with $\hat{p} \in (p_r, p_s)$. A borrower of type $\hat{p}$ earns $\overline{u}$ under C – if he earned more, then borrowers of type p in a neighborhood above $\hat{p}$ would also earn more than $\overline{u}$, since borrower payoffs are continuous in risk-type, which contradicts

[35] Such an AFLC exists and is unique. Note that the left-hand side of ZPC 4, call it LHS, is contained within $(0, \overline{p}R_s)$ for any admissible group contract; these bounds are because all interest rates r_k must be in $[0, R_s]$, and the set is open because not all interest rates can be equal under the unique homogeneous matching constraint. The set of admissible AFLCs is indexed by $r \in (0, R_s)$, and there exists an AFLC that can raise any amount in $(0, \overline{p}R_s)$. This is clear because as $r \to 0$, $LHS \to 0$; as $r \to R_s$, $LHS \to \overline{p}R_s$; and because the r_k's in an AFLC, and thus LHS too, are continuous in r. Thus if C can satisfy ZPC 4, an admissible AFLC also can. Uniqueness is clear because LHS is strictly increasing in r for $r \in (0, R_s)$.

[36] Note that

$$\begin{aligned}
&\frac{p_s\binom{n-1}{k-1}}{(1-p_s)\binom{n-1}{k}} > \frac{\overline{p^{n-k+1}(1-p)^{k-1}}\binom{n-1}{k-1}}{\overline{p^{n-k}(1-p)^{k}}\binom{n-1}{k}} \\
\iff &\int_{p_r}^{p_s} p_s p^{n-k}(1-p)^k f(p)dp > \int_{p_r}^{p_s} (1-p_s)p^{n-k+1}(1-p)^{k-1} f(p)dp \\
\iff &\int_{p_r}^{p_s} (p_s - p)p^{n-k}(1-p)^{k-1} f(p)dp > 0\ .
\end{aligned}$$

the set of borrowing types being $[p_r, \hat{p}]$.

Now by above logic, given set of borrowing types $[p_r, \hat{p}]$, the type-$\hat{p}$ borrowing payoff is uniquely maximized subject to the appropriate ZPC – identical to ZPC 4, except the expectation is conditional on $p \in [p_r, \hat{p}]$ – by the AFLC satisfying this ZPC at equality.[37] Thus, if C is not an AFLC, there exists an AFLC that gives the lender zero profits on borrowing types $[p_r, \hat{p}]$ and delivers a borrower of type $\hat{p}$ a payoff strictly greater than $\overline{u}$. But then, in this AFLC, a) agents of type p in a neighborhood above $\hat{p}$ would also earn strictly more than $\overline{u}$ by borrowing, by continuity, and thus choose to borrow; and b) the ZPC for this larger set of borrowers would be satisfied, by Lemma 1, since this AFLC satisfies the ZPC assuming a worse pool of borrowers, $[p_r, \hat{p}]$. Since all surplus from the loans to types $[p_r, \hat{p}]$ goes to the borrowers (guaranteed because the AFLC exactly breaks even over these types), and strictly positive surplus goes to some borrowers above $\hat{p}$, this proves that strictly greater borrower surplus is available with an AFLC.

The above arguments guarantee two things. First, since every contract satisfying all constraints is at least weakly dominated by an AFLC that satisfies all constraints, optimization may be restricted to AFLCs. A solution exists in this simplified optimization.[38] It is then clear that the set of optimal contracts is non-empty and always contains an AFLC.

Second, the argument in the final case above guarantees that the set of optimal contracts includes only AFLCs if some but not all agents borrow at the optimum. ■

Proof of Lemma 3. Assume a menu of group contracts C exists that satisfies all constraints, and such that the safest type C attracts to borrow is $\hat{p}$. That is, C delivers type $\hat{p}$ a borrowing payoff of at least $\overline{u}$, and satisfies limited liability and monotonicity of each contract in C; homogeneous matching uniquely induced by C; incentive compatibility guaranteeing that the contract intended for type p is preferred by a group of type p to all other contracts in C; and lender ZPC given that each group is homogeneous and borrows under its intended contract and that the set of borrowing types is $[p_r, \hat{p}]$, for some $\hat{p} \in [p_r, p_s]$.[39]

Let $(r_0^p, r_1^p, ..., r_{n_p-1}^p)$ be the contract in C intended for type p, and $P = (r_0^{\hat{p}}, r_1^{\hat{p}}, ..., r_{n_{\hat{p}}-1}^{\hat{p}})$. Note that at least one interest rate $r_k^{\hat{p}}$ in P must be strictly less than R_s. This is because if all interest rates in P equaled R_s, then by incentive compatibility all interest rates in every other contract would equal R_s; but if every contract had a constant interest rate, homogeneous matching would not be the unique outcome.

[37] Specifically, the logic applies after changing type from p_s to $\hat{p}$ when considering borrower payoffs.

[38] It is clear that the optimal AFLC is the one with the smallest interest rate r that satisfies all constraints, since this leaves most surplus to the borrowers (and uniquely so if there are borrowers). It remains to show that such a minimum r exists. To do so, let Z be the set of all r such that an AFLC involving r satisfies all constraints. Clearly, Z is the subset of $(0, R_s)$ such that an AFLC involving r satisfies the ZPC. First, Z is non-empty. This is by the assumption that there exists a contract C satisfying all constraints, and the main result above that in such a case, there is an AFLC satisfying all constraints and at least as good as any C that satisfies all constraints. Second, Z has a minimum. This is because $r < \rho$ could never satisfy the ZPC, so Z is a subset of $[\rho, R_s)$; and due to the continuity of the ZPC with respect to r and the ZPC's weak inequality. Together, these imply any downward limit point of Z is contained in Z.

[39] This is the set of borrowing types because every borrowing group of type $p \leq \hat{p}$ earns at least $\overline{u}$ per member with the $\hat{p}$-contract, by Lemma 1; and by incentive compatibility, they earn at least as much with their own contract as with the $\hat{p}$-contract.

Now, incentive compatibility of C implies that

$$\overline{R} - \sum_{k=0}^{n_p-1} p^{n_p-k}(1-p)^k \binom{n_p-1}{k} r_k^p \geq \overline{R} - \sum_{k=0}^{n_{\hat{p}}-1} p^{n_{\hat{p}}-k}(1-p)^k \binom{n_{\hat{p}}-1}{k} r_k^{\hat{p}} \ , \quad i.e.,$$

$$\sum_{k=0}^{n_{\hat{p}}-1} p^{n_{\hat{p}}-k}(1-p)^k \binom{n_{\hat{p}}-1}{k} r_k^{\hat{p}} \geq \sum_{k=0}^{n_p-1} p^{n_p-k}(1-p)^k \binom{n_p-1}{k} r_k^p \ , \quad \forall p \in [p_r, \hat{p}] \ .$$

Thus

$$\int_{p_r}^{\hat{p}} \left[\sum_{k=0}^{n_{\hat{p}}-1} p^{n_{\hat{p}}-k}(1-p)^k \binom{n_{\hat{p}}-1}{k} r_k^{\hat{p}} \right] f(p)dp \geq \int_{p_r}^{\hat{p}} \left[\sum_{k=0}^{n_p-1} p^{n_p-k}(1-p)^k \binom{n_p-1}{k} r_k^p \right] f(p)dp \geq \rho \ .$$

The second inequality here is exactly the ZPC satisfied by C, and the first follows from incentive compatibility (the inequality above it). This guarantees that the ZPC satisfied by C is satisfied if only the single contract P is offered.

We have thus shown that there exists a pooling contract (P) that delivers the $\hat{p}$-borrower at least $\overline{u}$, satisfies limited liability and monotonicity, has at least one interest rate strictly less than R_s, and satisfies the ZPC given that each group is homogeneous and that the set of borrowing types is $[p_r, \hat{p}]$. Further, logic of the Proof of Proposition 1 makes clear that if so, there is an AFLC, call it P', that delivers the $\hat{p}$-borrower at least $\overline{u}$, satisfies limited liability and monotonicity, involves $r < R_s$, and satisfies with equality the ZPC given that each group is homogeneous and that the set of borrowing types is $[p_r, \hat{p}]$. This is because an AFLC that satisfies the given ZPC at equality provides at least as high a $\hat{p}$-payoff as any pooling contract subject to limited liability, monotonicity, and the given ZPC; and $r < R_s$ in P', since otherwise P' would give the $\hat{p}$-borrower a lower payoff than P.

The next step is to show that P' satisfies the remaining constraints and delivers outreach of at least $[p_r, \hat{p}]$. Outreach of at least $[p_r, \hat{p}]$ follows from Lemma 1 and the fact that P' delivers type $\hat{p}$ at least $\overline{u}$. P' also guarantees homogeneous matching, by Lemma 2, since it involves $r < R_s$ and $r > 0$ (otherwise it could not satisfy C's ZPC). Finally, P' allows the lender to break even, since it breaks even on borrowers of types in $[p_r, \hat{p}]$, and therefore over the set of actual borrowers, which is $[p_r, \hat{p}']$ for some $\hat{p}' \geq \hat{p}$, by Lemma 1.

We have thus shown that if a menu of contracts achieves outreach $[p_r, \hat{p}]$, there exists a pooling AFLC that causes the lender to exactly break even over this set of borrowing types and also achieves at least this outreach. It remains to show this AFLC can achieve as much borrower surplus as the menu. This is guaranteed because all surplus from loans to types $[p_r, \hat{p}]$ goes to the borrowers in this AFLC, because the AFLC exactly breaks even over these types. ■

Proof of Proposition 2. The Proof uses notation and the CLT Corollary from the beginning of the Appendix.

Note that in an AFLC with group size n and interest rate r, lender profits assuming all borrow (the left-hand side of ZPC 4) are continuous and strictly increasing in r; strictly exceed ρ when $r = R_s$, given that $\mathcal{G} > p_s/p_r$; and are strictly less than ρ when $r = \rho$, since

$r = \rho$ breaks even only if there is no default. This implies there exists an $r \in (\rho, R_s)$ that exactly satisfies ZPC 4; call it r_n.

The borrowing payoff of an agent of type p_s under AFLC with r_n can be written

$$\overline{R} - r_n B_{n,n-k^*(n,r_n)+1,p_s} - R_s p_s (1 - B_{n-1,n-k^*(n,r_n),p_s}) \; . \tag{11}$$

This is because when the number of failures in the group is less than k^* (successes greater than $n - k^*$), the group pays nr_n and thus each borrower pays r_n (in expectation); and that iff the borrower succeeds *and* at least k^* of the other $n-1$ group members fail (less than $n - k^*$ succeed), the borrower pays R_s. The ZPC for an AFLC with arbitrary r assuming all agents borrow can be written analogously:

$$r \cdot \overline{B_{n,n-k^*(n,r)+1,p}} + R_s \cdot \overline{p(1 - B_{n-1,n-k^*(n,r),p})} \geq \rho \; . \tag{12}$$

Let $\mu \equiv \mathcal{G}/(p_s/p_r)$; by Assumption A3, $\mu > 1$. We first show that $\lim_{n\to\infty} r_n = \rho$. Sufficient for this is that for any $\kappa \in (1, \mu)$ and n high enough, $r = \kappa\rho$ satisfies ZPC 12 with strict inequality. For, this implies $\lim_{n\to\infty} r_n < \kappa\rho$ for any $\kappa \in (1, \mu)$; and, $r_n > \rho$. Fix $\kappa \in (1, \mu)$ and let $r = \kappa\rho$. By inspection of ZPC 12, the claim is proved if $\lim_{n\to\infty} \overline{B_{n,n-k^*(n,\kappa\rho)+1,p}} = 1$. Since $\overline{B_{n,n-k^*(n,\kappa\rho)+1,p}} \geq B_{n,n-k^*(n,\kappa\rho)+1,p_r}$, also sufficient is $\lim_{n\to\infty} B_{n,n-k^*(n,\kappa\rho)+1,p_r} = 1$, which is guaranteed by the CLT Corollary since $\kappa < \mu$.

Given that $\lim_{n\to\infty} r_n = \rho$, the CLT Corollary guarantees that $\lim_{n\to\infty} B_{n,n-k^*(n,r_n)+1,p_s} = 1$. Since $B_{n-1,n-k^*(n,r_n),p_s} \geq B_{n,n-k^*(n,r_n)+1,p_s}$, this implies that $\lim_{n\to\infty} B_{n-1,n-k^*(n,r_n),p_s} = 1$. Thus the entire payoff 11 approaches $\overline{R} - \rho$. $\mathcal{N} > 1$ implies that $\overline{R} - \rho > \overline{u}$. Thus, the safest borrower, and hence all borrowers by Lemma 1, choose to borrow if n is large enough and the lender offers an AFLC with r_n. The lender exactly breaks even with r_n, implying also that borrower surplus is maximal. ■

Proof of Lemma 4. Under random matching, every borrower has the same expected payment due when he succeeds, since the distribution over realized partners and their outcomes is the same for all, and the contract faced and its complete affordability are also the same for all. Call this expected payment upon success X. Then borrower i's payoff to borrowing is $\overline{R} - p_i X$, only agents with $p_i \leq \hat{p} \equiv (\overline{R} - \overline{u})/X$ borrow, and the lender ZPC is $X \cdot E(p|p \leq \hat{p}) \geq \rho$. Limited liability implies $X \leq R_s$. It is clear that an individual loan with $r = X$ results in the same expected payoffs for borrowers and satisfies the same constraints. Thus, any borrower payoffs accomplishable by a permissible group contract resulting in X can be accomplished by a permissible individual loan contract involving $r = X$. ■

Proof of Lemma 5. The proof uses notation defined in the beginning of the Appendix.

Fix $n \geq 2$ and define EC_m as the expected payment due from a group choosing m risky projects and $n - m$ safe projects, and G_m as a vector of n risk-types with m equaling p_r and $n - m$ equaling p_s. Then, as discussed in the Proof of Lemma 2, under an AFLC with interest rate r,

$$EC_m = nrB_{n,n-k^*+1,G_m} + \sum_{j=k^*}^{n} (n-j) R_s \tilde{B}_{n,n-j,G_m} \; .$$

The general group IC constraint can be written

$$n\overline{R}_s - EC_0 \geq (n-m)\overline{R}_s + m\overline{R}_r - EC_m \ , \qquad i.e.\ , \qquad \mathcal{N} \geq \frac{1}{\rho}\frac{EC_0 - EC_m}{m} \ ,$$

for $m \in \{1, 2, ..., n\}$. The first inequality guarantees the group payoff for all safe projects exceeds the group payoff for m risky projects and the rest safe; the second rearranges the first. If the right-hand side of the second inequality is increasing in m, then IC constraint 6 ($m = n$) implies all the others. Sufficient for this is that, for $m \in \{1, 2, ..., n-1\}$,

$$\frac{EC_0 - EC_{m+1}}{m+1} > \frac{EC_0 - EC_m}{m} \ , \qquad i.e.\ , \qquad (m+1)EC_m > EC_0 + mEC_{m+1} \ .$$

We argue in two steps that the second inequality is true. First, note that the function $EC(p_1, p_2, ..., p_n)$ defined over n risk-types in $(0, 1)$, i.e. with domain $(0, 1)^n$, is strictly submodular. This follows from the Proof of Lemma 2, where we showed that $n\overline{R} - EC_G$ is strictly supermodular under an AFLC with $r \in (0, R_s)$, G being a vector of n risk-types in $(0, 1)$; this implies EC is strictly submodular. Second, we argue that strict submodularity of EC implies the inequality. To see this, note that for any integers m and m', with $m \in \{1, 2, ..., n-1\}$ and $m' \in \{1, 2, ..., m\}$,

$$\begin{aligned} EC_{m'} + EC_m &= EC(\underbrace{p_r, ..., p_r}_{m'-1}, \underbrace{p_s, ..., p_s}_{n-m'}, p_r) + EC(\underbrace{p_r, ..., p_r}_{m}, \underbrace{p_s, ..., p_s}_{n-m}) \\ &> EC(\underbrace{p_r, ..., p_r}_{m'-1}, \underbrace{p_s, ..., p_s}_{n-m'+1}) + EC(\underbrace{p_r, ..., p_r}_{m}, \underbrace{p_s, ..., p_s}_{n-m-1}, p_r) = EC_{m+1} + EC_{m'-1} \ , \end{aligned}$$

where the equalities come from symmetry of EC (i.e. invariance to permutation of arguments) and the inequality from strict submodularity of EC. Applying this fact m times to $(m+1)EC_m$ gives the desired inequality.[40] ■

Proof of Proposition 3. The Proof uses notation and the CLT Corollary from the beginning of the Appendix.

By an argument analogous to the one from the Proof of Proposition 2, we can define r_n as the interest rate in an AFLC with group size n that causes the lender to exactly break even, assuming the group chooses all safe projects. As in that Proof, we know $r_n \in (\rho, R_s)$, and by an analogous argument to the one in that Proof, $\lim_{n\to\infty} r_n = \rho$.

If the lender offers an AFLC with r_n, it exactly breaks even; hence, borrowers in groups all choosing safe projects earn first-best payoffs $\overline{R}_s - \rho$, greater than $\overline{u}$ by assumption. Thus

[40] Note that

$$\begin{aligned} (m+1)EC_m &= (m-1)EC_m + EC_m + EC_m & &> (m-1)EC_m + EC_{m-1} + EC_{m+1} \\ &= (m-2)EC_m + EC_{m-1} + EC_m + EC_{m+1} & &> (m-2)EC_m + EC_{m-2} + 2\,EC_{m+1} \\ = \ldots > \ldots &= (1)\ EC_m + EC_2 + EC_m + (m-2)EC_{m+1} & &> (1)\ EC_m + EC_1 + (m-1)EC_{m+1} \\ &= EC_1 + EC_m + (m-1)EC_{m+1} & &> EC_0 + mEC_{m+1} \ . \end{aligned}$$

borrowers prefer all safe projects to the outside option. By Lemma 5, it remains to show they prefer all safe projects to all risky projects if the lender offers an AFLC with r_n; i.e., writing payoffs as described in the Proof of Proposition 2, to show that

$$\begin{aligned}
&\overline{R}_s - r_n B_{n,n-k^*(n,r_n)+1,p_s} - R_s p_s(1 - B_{n-1,n-k^*(n,r_n),p_s}) \geq \\
&\quad \overline{R}_r - r_n B_{n,n-k^*(n,r_n)+1,p_r} - R_s p_r(1 - B_{n-1,n-k^*(n,r_n),p_r})\ , \\
&\textit{i.e.,} \\
&\mathcal{N} \geq \{r_n[B_{n,n-k^*(n,r_n)+1,p_s} - B_{n,n-k^*(n,r_n)+1,p_r}] + \\
&\quad R_s[p_s(1 - B_{n-1,n-k^*(n,r_n),p_s}) - p_r(1 - B_{n-1,n-k^*(n,r_n),p_r})]\}\ /\ \rho
\end{aligned}$$

Since $\mathcal{N} > 0$ and $\lim_{n\to\infty} r_n = \rho$, this holds for n large enough if $\lim_{n\to\infty} B_{n,n-k^*(n,r_n)+1,p_r} = 1$ and $\lim_{n\to\infty} B_{n-1,n-k^*(n,r_n),p_s} = 1$. The CLT Corollary guarantees the first of these limits, which in turn guarantees the second limit, since $B_{n-1,n-k^*(n,r_n),p_s} \geq B_{n-1,n-k^*(n,r_n),p_r}$ and $B_{n-1,n-k^*(n,r_n),p_r} \geq B_{n,n-k^*(n,r_n)+1,p_r}$. ■

Proof of Propositions 4 and 5. The Proof uses notation defined in the beginning of the Appendix, and the CLT Fact stated there.

We first prove Claim A of Proposition 5. Fix $\epsilon \in (0,1)$ and $\overline{L}' \in (\overline{L}, 1]$ such that $\overline{L}'(1 - \gamma p/\rho) < \overline{L}$, and define

$$\kappa \equiv \min\left\{ \frac{(\overline{K}/\overline{L}' + \gamma)p/\rho - 1}{2}\ ,\ \frac{\overline{R}/\rho - 1}{2}\ ,\ \frac{\epsilon(1-p)}{2p} \right\}\ .$$

In part because $\overline{R} > \rho$ (implied by BSW's assumptions), one can verify that $\kappa > 0$. Consider full liability size-n group contracts with interest rate $r = (1+\kappa)\rho$, where a borrower of loan size L is sanctioned $\overline{K}L/\overline{L}'$ if the entire group obligation (nrL) is not repaid. We wish to show that for n high enough, this contract satisfies lender break-even (respecting incentive compatibility and limited liability) for all loan sizes $L \in (0, \overline{L}']$, and provides borrowers of all wealth levels $w \in [1 - \overline{L}', 1)$ with payoffs of at least $\overline{R} - \rho - \epsilon\rho(1-w)(1-p)/p$.

First note that the sanction $\overline{K}L/\overline{L}'$ is feasible for all $L \in (0, \overline{L}']$. Define

$$j^*(n) \equiv \max\left\{ \left\lceil \frac{pn(1+\kappa)}{(\overline{K}/\overline{L}' + \gamma)p/\rho} \right\rceil\ ,\ \left\lceil \frac{pn(1+\kappa)}{\overline{R}/\rho} \right\rceil \right\}\ .$$

Sufficient for full group repayment of nrL to be feasible and incentive compatible for any loan size $L \in (0, \overline{L}']$ if the number of group successes is at least $j^*(n)$ is

$$\frac{nrL}{j^*(n)} \leq \overline{K}L/\overline{L}' + \gamma \qquad \textit{and} \qquad \frac{nrL}{j^*(n)} \leq R\ .$$

Plugging in for r and $j^*(n)$, it is straightforward to show these hold for any $L \in (0, 1]$. Thus the repayment rate for all loan sizes $L \in (0, \overline{L}']$ is at least $B_{n,j^*(n),p}$.

Note that CLT Fact (10) guarantees $\lim_{n\to\infty} B_{n,j^*(n),p} = 1$, since

$$p > \lim_{n\to\infty} \frac{j^*(n)}{n} = \max\left\{ \frac{p(1+\kappa)}{(\overline{K}/\overline{L}' + \gamma)p/\rho} \; , \; \frac{p(1+\kappa)}{\overline{R}/\rho} \right\} \; .$$

Thus there exists an $\overline{n}$ such that $n \geq \overline{n}$ implies

$$B_{n,j^*(n),p} \geq \max\left\{ \frac{1}{1+\kappa} \; , \; 1 - \frac{\epsilon(1-p)}{2p}\frac{\rho\overline{L}'}{\overline{K}} \right\} \; .$$

The lender breaks even for $L \in (0, \overline{L}']$ if $n \geq \overline{n}$, since for $r = (1+\kappa)\rho$,

$$B_{n,j^*(n),p} \geq \frac{1}{1+\kappa} \qquad \Longrightarrow \qquad B_{n,j^*(n),p} \cdot rL \geq \rho L \; .$$

Finally we show that for $n \geq \overline{n}$, borrower payoffs for all wealth levels $w \in [1-\overline{L}', 1)$ and loan size $L = 1-w$ are at least $\overline{R} - \rho - \rho(1-w)\epsilon(1-p)/p$. Let $n \geq \overline{n}$. For some repayment rate $q(n,w)$, a borrower of wealth w earns

$$\overline{R} - \rho w - q(n,w)rL - [1-q(n,w)]\frac{\overline{K}L}{\overline{L}'} = \overline{R} - \rho - \rho(1-w)[\kappa + (1-q(n,w))(\frac{\overline{K}}{\rho\overline{L}'} - 1 - \kappa)]$$

$$\geq \overline{R} - \rho - \rho(1-w)[\kappa + (1-q(n,w))\frac{\overline{K}}{\rho\overline{L}'}] \geq \overline{R} - \rho - \rho(1-w)[\kappa + (1-B_{n,j^*(n),p})\frac{\overline{K}}{\rho\overline{L}'}] \; ,$$

where the equality comes from plugging in for r and L and rearranging, and the second inequality is because $q(n,w) \geq B_{n,j^*(n),p}$ for all $w \in [1-\overline{L}', 1)$. The claim is established since

$$(1-B_{n,j^*(n),p})\frac{\overline{K}}{\rho\overline{L}'} \leq \frac{\epsilon(1-p)}{2p} \; , \qquad so \qquad \kappa + (1-B_{n,j^*(n),p})\frac{\overline{K}}{\rho\overline{L}'} \leq \frac{\epsilon(1-p)}{p} \; .$$

Proposition 4 is proved identically, except that $\overline{L}'$ is fixed in the interval $(0, \overline{L})$ and γ is set to zero in all expressions. Claim B of Proposition 5 is also proved identically after fixing $\overline{L}' = 1$. ■

Proof of Proposition 6. Assumption $\mathcal{G} > p_s/\overline{p}$ guarantees that ZPC 8 is satisfied by an AFLC with $n = 2$ and $r_0 < R_s$; thus there exists an (unique) AFLC with $n = 2$ that satisfies ZPC 8 with equality, and $r_0 < r_1$ in this AFLC. Consider this AFLC, define $\delta \equiv r_1 - r_0$ (> 0) and $\tilde{\tilde{p}}_s \equiv \pi^2 p_s + (1-\pi^2)\overline{p}$, and let π denote $\pi(2) (> 0)$. It is straightforward to verify that homogeneous matching obtains.[41] ZPC 8 at equality becomes

$$\overline{\tilde{p}^2} r_0 + \overline{\tilde{p}(1-\tilde{p})}(r_0+\delta) = \rho \ , \qquad i.e., \qquad r_0 = \frac{\rho - \overline{\tilde{p}(1-\tilde{p})}\delta}{\overline{p}} \ .$$

Using payoff expression 7, the condition for including the safest ("safe") borrowers, and thus achieving perfect outreach, is

$$\overline{R} - p_s \tilde{\tilde{p}}_s r_0 - p_s(1-\tilde{\tilde{p}}_s)(r_0+\delta) \geq \overline{u} \ , \qquad i.e.,$$

$$\mathcal{N} \geq \frac{p_s[r_0 + (1-\tilde{\tilde{p}}_s)\delta]}{\rho} = \frac{p_s}{\overline{p}}[1 - (\delta/\rho)\pi^2 \overline{p(p_s - p)}] \equiv \widetilde{\mathcal{N}} \ ,$$

where the last equality substitutes in for r_0 from the ZPC and rearranges. Thus full efficiency is attained by group lending with groups of size 2 for $\mathcal{N} \geq \widetilde{\mathcal{N}}$. By inspection, $\widetilde{\mathcal{N}} < p_s/\overline{p}$

For general n, including safe borrowers requires the following necessary condition, using expression 7:

$$\overline{R} - p_s \sum_{k=0}^{n-1} [1-\pi(n)]\, \overline{\tilde{p}^{n-k-1}(1-\tilde{p})^k} \binom{n-1}{k} r_k \geq \overline{u} \ .$$

Note from ZPC 8 that

$$\sum_{k=0}^{n-1} [1-\pi(n)] \overline{\tilde{p}^{n-k-1}(1-\tilde{p})^k} \binom{n-1}{k} r_k \geq \frac{\rho - \pi(n)\sum_{k=0}^{n-1} \overline{p\tilde{p}^{n-k-1}(1-\tilde{p})^k} \binom{n-1}{k} r_k}{\overline{p}} \equiv \frac{\rho - Q_n}{\overline{p}} \ ,$$

say. So, necessary for fully efficient lending is

$$\overline{R} - p_s \frac{\rho - Q_n}{\overline{p}} \geq \overline{u} \ , \qquad i.e., \qquad \mathcal{N} \geq \frac{p_s}{\overline{p}}\left(1 - \frac{Q_n}{\rho}\right) \ .$$

Now $\lim_{n\to\infty} Q_n = 0$, because $Q_n \geq 0$, $\lim_{n\to\infty} \pi(n) = 0$, and

$$Q_n \equiv \pi(n) \sum_{k=0}^{n-1} \overline{p\tilde{p}^{n-k-1}(1-\tilde{p})^k} \binom{n-1}{k} r_k \leq \pi(n) R_s \sum_{k=0}^{n-1} \overline{\tilde{p}^{n-k-1}(1-\tilde{p})^k} \binom{n-1}{k} = \pi(n) R_s \ ,$$

the inequality in part because $r_k \leq R_s$. Applying this to the condition for efficient lending establishes that for any $\mathcal{N} < p_s/\overline{p}$, fully efficient lending is not achieved if n is high enough.

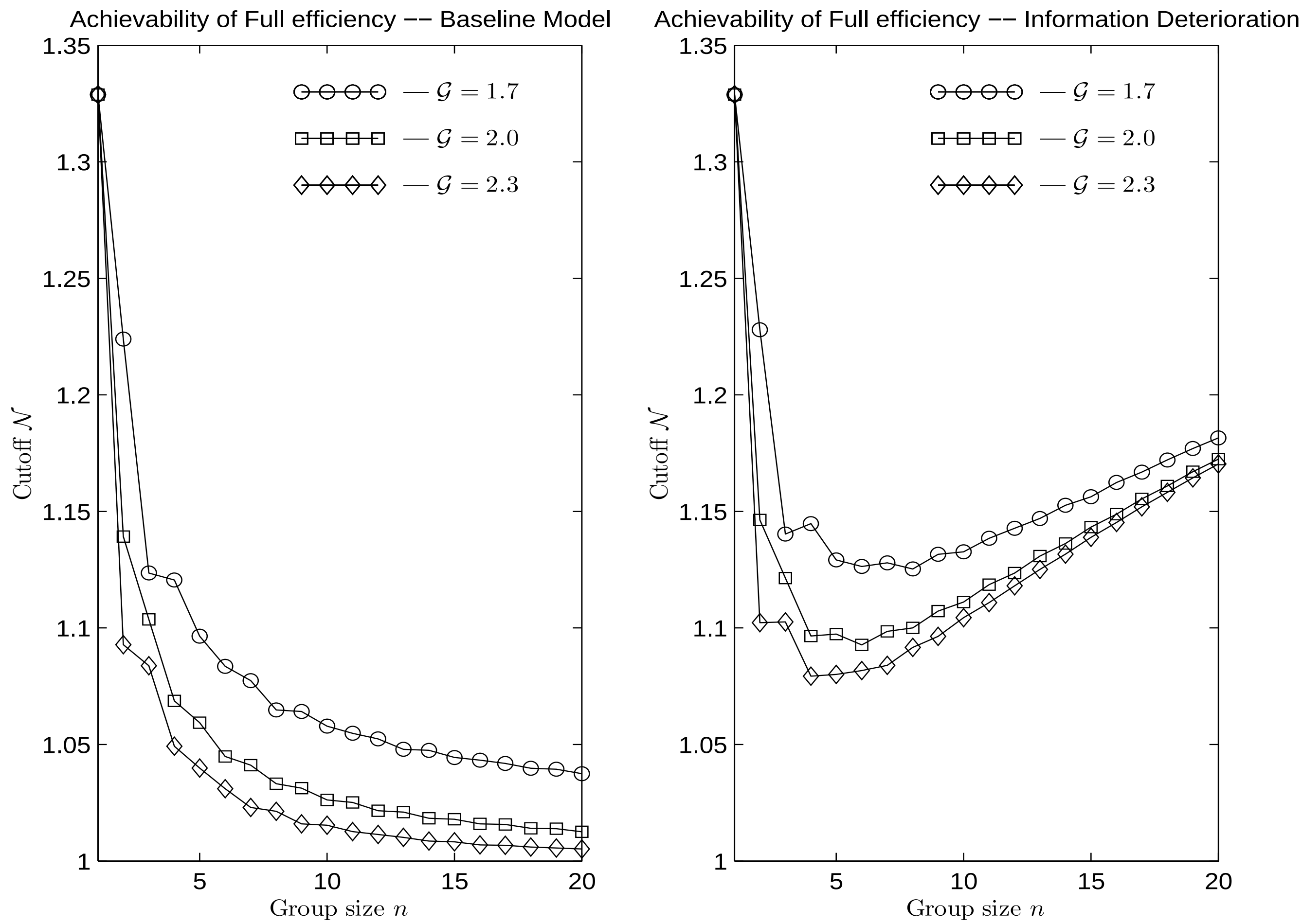
In sum, since $\widetilde{\mathcal{N}} < p_s/\overline{p}$, there exists an $\overline{n} > 2$ such that for $n \geq \overline{n}$, fully efficient lending is not achieved if $\mathcal{N} < (\widetilde{\mathcal{N}} + p_s/\overline{p})/2$; fully efficient lending is not achieved by individual lending if $\mathcal{N} < p_s/\overline{p}$ (as shown in section 5); but fully efficient lending is achieved by group lending with $n = 2$ if $\mathcal{N} \geq \widetilde{\mathcal{N}}$. Since $\widetilde{\mathcal{N}} < (\widetilde{\mathcal{N}} + p_s/\overline{p})/2 < p_s/\overline{p}$, this establishes the result. ■

[41] The expected group payoff function for a group of (pre-replacement) types p_i and p_j is

$$2\overline{R} - p_i \tilde{\tilde{p}}_j r_0 - p_i(1-\tilde{\tilde{p}}_j) r_1 - p_j \tilde{\tilde{p}}_i r_0 - p_j(1-\tilde{\tilde{p}}_i) r_1$$

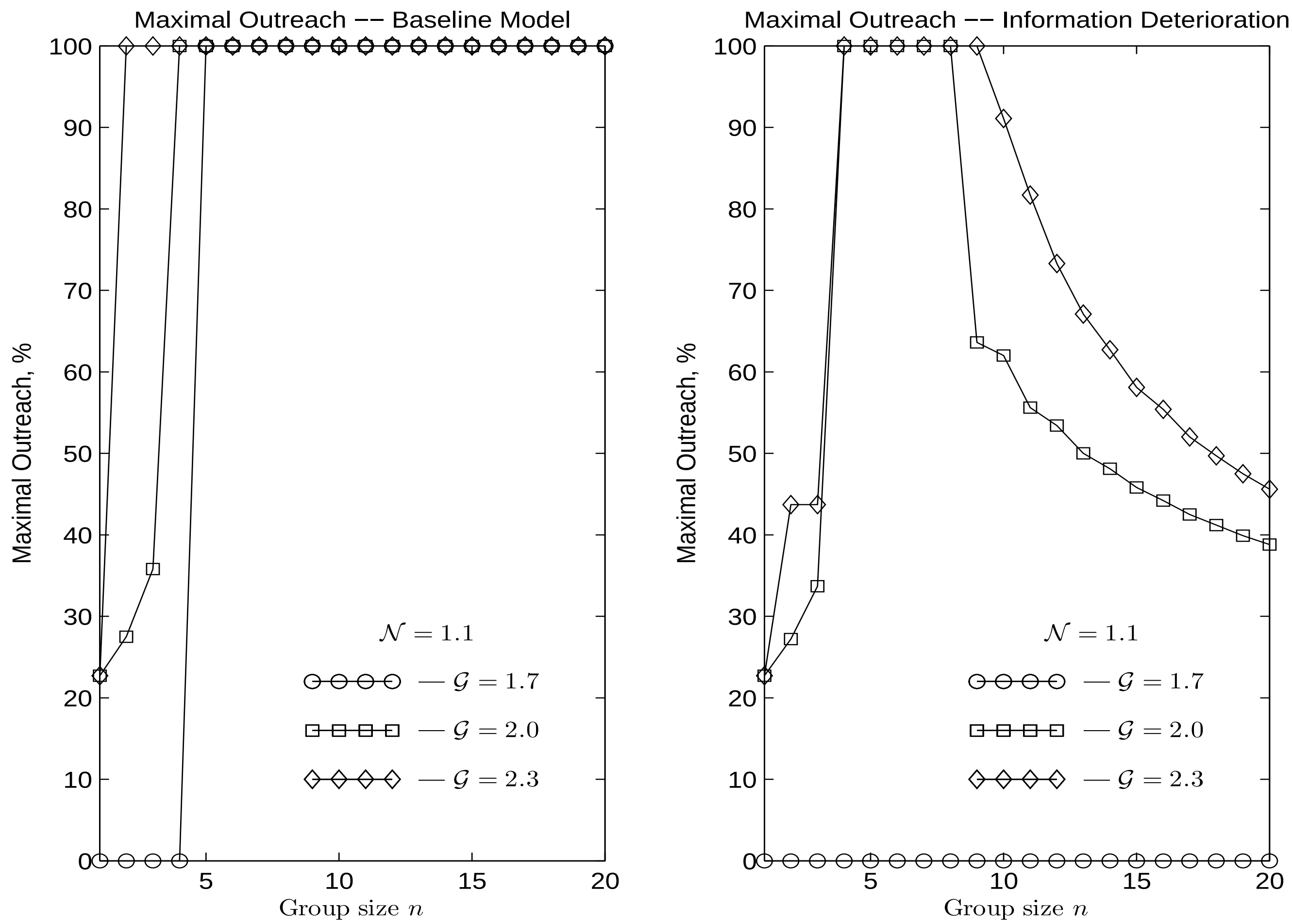
and the cross-partial with respect to p_i and p_j is $2\delta\pi^2 > 0$.

Figure 1: When is Full Efficiency Achievable, as a Function of Group Size



Note: For three values of gross excess return $\mathcal{G}$, this graphs against group size n the cutoff value for net excess return $\mathcal{N}$, above which fully efficient lending and complete outreach are attainable. Risk types are distributed uniformly over $[0.5, 0.99]$, and lender required return is $\rho = 1.1$. The left panel is the baseline model. The right panel is the information deterioration model of Section 7.1 with $\pi(n) = e^{0.02(1-n)}$.

Figure 2: Outreach (Efficiency) as a Function of Group Size



Note: For three values of gross excess return $\mathcal{G}$, this graphs against group size n the maximum outreach (efficiency) achievable at $\mathcal{N} = 1.1$. Risk types are distributed uniformly over $[0.5, 0.99]$, and lender required return is $\rho = 1.1$. The left panel is the baseline model. The right panel is the information deterioration model of Section 7.1 with $\pi(n) = e^{0.02(1-n)}$.