On Convergence in the Spatial AK Growth Models*

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Abstract

Recent research in economic theory attempts to study optimal economic growth and spatial location of economic activity in a unified framework. So far, the key result of this literature - asymptotic convergence, even in the absence of decreasing returns to capital - relies on specific assumptions about the objective of the social planner. We show that this result does not depend on such restrictive assumptions and obtains for a broader class of objective functions. We also generalize this finding, allowing for the time-varying technology parameter, and provide an explicit solution for the dynamics of spatial distribution of the capital stock.

Keywords: Economic Growth; Convergence; Spatial Dynamics; Partial Differential Equations.

JEL codes: C60, O40, O11, R11.

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1 Introduction

Two major fields of research in theoretical economics - theory of economic growth and models of new economic geography - have been developing separately and independently until recently. The need to integrate them in a common framework, as well as the underlying fundamental mechanisms between endogenous growth and agglomeration of economic activity - in particular, increasing returns to scale - was forcefully stated by Lucas (1988) and, in more detail, by Krugman (1997). Over the last decade, economists have been trying to construct such unified analytical frameworks that allow to capture and describe the evolution of economic activity both in space and in time. Most of this recent literature has been summarized in a survey by Desmet and Rossi-Hansberg (2010), as well as in Chapters 18 and 19 of Acemoglu (2008). What emerges from these surveys is that while a commonly agreed framework has not yet emerged, research has been evolving using several different and highly promising approaches.

One of these approaches is to build optimal growth models with spatial dimension allowing for continuous space and time, in which spatio-temporal evolution of capital stock is described by a partial differential equation, usually of the diffusion (parabolic) type. This framework, initially suggested by Isard and Liossatos (1979), has been rigorously developed recently by Brito (2004) and Boucekkine et al. (2009), for the spatial version of Ramsey-Cass-Koopmans optimal growth model (i.e. with decreasing returns to capital) and by Boucekkine et al. (2013) for an AK growth model.

The major result in all of these papers is the asymptotic disappearance of spatial inequality, i.e. convergence of the capital stock over time to the same level in all the regions, despite the initially heterogeneous spatial distribution of capital. This is not too surprising for the spatial Ramsey-Cass-Koopmans model, given the decreasing returns to capital, and thus an intuitive economic reason for the flow of capital from more capital-abundant to capital-scarce regions. However, the convergence result is much more surprising for the spatial AK endogenous growth model, given that the returns to capital are constant; it is well known that the non-spatial versions of the AK model exhibit non-convergence (Romer 1986, Lucas 1988, Rebelo 1991).

Boucekkine et al. (2013) rely on a specific assumption on the objective of the benevolent social planner: she treats equally all the individuals, independently of their location (and thus initial endowment of capital). Given this, the social planner chooses to smooth the (detrended) consumption, both across space and time. This assumption could be difficult to justify according to certain ethical criteria, for example, giving more weight to individuals located in areas with a lower initial level of capital stock. Thus, a natural question arises: How general is the convergence result in the spatial AK model? In particular, does this surprising result rely on the particular assumptions concerning the objective function of the benevolent social planner?

This note provides an answer to this question and generalizes the findings of Boucekkine et al. (2013) in two important ways. First, we find that the convergence result does not rely on restrictive assumptions on the objective of the social planner. In particular, we show that the asymptotic convergence result holds for a program of the social planner with any objective function which gives rise to a continuous consumption function, provided that the present discounted value of the flow of maximum consumption (on the entire space) does not exceed the initial capital stock at any point of the space.

Second, we show that these results hold for the AK production function in which technology (A) evolves over time. This is an important generalization, because technological change can potentially alter the spatial dynamics of capital stock, and thus a priori it is unclear whether the convergence result would hold in such settings.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 develops our analytical results. Section 4 highlights directions for future research and concludes. Technical proofs are relegated to the Appendix.

2 Model

In modelling the spatial economic growth process, we follow the approach initiated by Brito (2004) and further developed by Boucekkine et al. (2009) and Boucekkine et al. (2013).

Spatial dimension is modelled as a circle of radius one, on which atomistic economic agents are assumed to be uniformly distributed. Thus, letting x to denote the geographic location of an agent, we have $x \in \mathbb{T} = [0, 2\pi]$, where boundary points of $\mathbb{T}$ coincide. This circle is a stylized representation of different regions in a country (in case of a closed economy), or - allowing for perfect capital mobility - can represent the global economy. Time is assumed to be continuous and evolves from zero to infinity: $t \in \mathbb{R}^+ = [0, \infty)$.

Our main object of interest is the spatial distribution of capital stock and its evolution over time. We denote the capital accumulated at moment t in point x with $k(x, t)$. The initial distribution of capital stock $k(x,0)$ on the circle is a known function $k_0(x)$.

The production function is of AK-type (see Acemoglu, 2008, chapter 11, for detailed analysis of the properties of the non-spatial growth models with the AK aggregate production function), i.e. the returns to capital are constant. Importantly, we allow for technology to change over time (although the technological frontier in any given moment is the same in every point of the circle). Thus, $A(x,t) = A(t)$ denotes the level of technology at time t in the whole space.

The instantaneous budget constraint of the agent located at point x at moment t is

$$y(x,t) = c(x,t) + \tau(x,t) + s(x,t), \tag{1}$$

and simply states that the production $y(x,t)$ is divided between consumption $c(x,t)$, trade balance $\tau(x,t)$ (given that each region is a small open economy), and saving $s(x,t)$. Saving represents capital accumulation into the next instant of time:

$$s(x,t) = k_t(x,t).$$

As stated above, production uses the linear AK-technology:

$$y(x,t) = A(t)k(x,t).$$

Finally, concerning the trade balance, we assume that there is perfect capital mobility. In other words, consider a region (arc) R of the circle. The consumption in excess of unsaved output minus consumption of "domestic" output (i.e. of output produced within the region) comes from other regions, which is reflected by the trade balance of this region. However, in the balance of payments of the region R, this excess consumption has to be financed by capital outflows. Thus, the regional trade balance simply equals the symmetric of the inflow of capital from one of its border minus the outflow from the other border:

$$\int_R \tau(x,t)dx = -[k_x(b,t) - k_x(a,t)], \tag{2}$$

where b and a are the boundaries of the region R. Using the Fundamental Theorem of Calculus, the trade balance can thus be written as

$$\int_R \tau(x,t)dx = -\int_R k_{xx}(x,t)dx,$$

which, for a length of the region R tending to zero, simply becomes

$$\tau(x,t) = -k_{xx}(x,t).$$

Thus, the instantaneous budget constraint (1) can be written as the following equation of motion of capital:

$$A(t)k(x,t) = c(x,t) - k_{xx}(x,t) + k_t(x,t), \tag{3}$$

and this constraint must hold for any point x and moment t.

Moreover, given that we represent the space as a circle, the values of the capital stock must coincide at the endpoints of the interval $\mathbb{T} = [0, 2\pi]$, and the smooth-pasting condition must also hold, at any moment t:

$$k(0,t) = k(2\pi,t) \text{ and } k_x(0,t) = k_x(2\pi,t). \tag{4}$$

The problem of optimal growth in this economy is that of a social planner that maximizes a certain objective function $J(k_0, c(x,t))$ by choosing the consumption function $c(x,t)$, subject to the instantaneous budget constraint (3), the boundary value conditions (4), and the initial value condition $k(x,0) = k_0(x)$. Clearly, the value of the capital stock $k(x,t)$ must be non-negative everywhere and in any moment of time. Formally, this accounts to:

Problem 1 *Find a* non-negative classical solution*, namely a continuous function in the closed domain* $\bar{\Omega}$*, where* $\Omega = \mathbb{T} \times \mathbb{R}^+$*, twice-continuously differentiable with respect to* x *in* Ω*, of the linear parabolic partial differential equation*

$$k_t = k_{xx} + A(t)k(x,t) - c(x,t) \quad \forall (x,t) \in \Omega\,, \tag{5}$$

that satisfies the initial condition

$$k(x,0) = k_0(x), \quad \forall x \in \mathbb{T}\,. \tag{6}$$

The problem of the social planner is a highly complicated infinite-dimensional optimal control problem, where complications essentially arise because one of the constraints is in the form of a partial differential equation. In a key contribution, Boucekkine et al. (2013) develop an analytical methodology that allows to overcome this challenge by adapting the dynamic programming methods to this infinite-dimensional problem. However, they need to impose a specific form on the objective function of the social planner, in order to obtain a characterization of the optimal consumption function $c(x,t)$.

Instead, we attack this problem differently. We study the problem of finding a non-negative classical solution of the partial differential equation describing the equation of motion of capital stock, for a general (continuous) consumption function $c(x,t)$. In doing so, we determine two different sufficient conditions on the consumption function that guarantee the uniqueness and non-negativity of the explicit solution of the PDE problem. The first condition leads to a space-invariant consumption function (and is thus equivalent to the one posited by the objective function of the social planner in Boucekkine et al. 2013). However, the second condition is more general, and has a different economic interpretation. Next, we show that the asymptotic properties of the solution are similar to the ones determined by Boucekkine et al. (2013); in particular, we prove the convergence of the capital stock in every point of the circle to the same level as $t \to \infty$. Crucially, the second sufficient condition on the consumption function that we find is considerably weaker than those of Boucekkine et al. (2013). We thus show that the convergence result in the spatial AK model is not driven by a particular objective function of the social planner.

3 Analysis

The aim of this section is twofold: (i) to provide solutions (Theorems 1 and 2) of Problem 1 under two different assumptions on the consumption function $c(x, t)$, and (ii) to study the asymptotic behavior of such solutions (Propositions 6 and 7).

Assuming the spatio-temporal consumption function to be a smooth concave function with respect to the spatial variable for any positive time moment, the following result holds:

Theorem 1 *Let $\Omega = \mathbb{T} \times \mathbb{R}^+$. Assume the functions $A : \mathbb{R}^+ \to \mathbb{R}^+$, $k_0 : \mathbb{T} \to \mathbb{R}^+ \cup \{0\}$ and $c : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ are continuous in their respective domains. Assume also that*

$$c_{xx}(x, t) \leq 0 \quad \forall (x, t) \in \mathbb{T} \times \mathbb{R}^+ \,, \tag{7}$$

and

$$k_0(x) \geq \int_0^t e^{-\int_0^s A(z)\, dz} c(x, s)\, ds \quad \forall (x, t) \in \Omega \,. \tag{8}$$

Then the Problem 1 admits a unique non-negative classical solution.

Note that given the coincidence of the endpoints of the interval $\mathbb{T} = [0, 2\pi]$ and the smooth-pasting condition (4), the concavity assumption (7) leads to a consumption function that is invariant in space. Thus, it is equivalent to the one posited by the objective function of the social planner in Boucekkine et al. (2013). On the one hand, this is re-assuring, as (together with the convergence results in Section 3.2) it shows that the two approaches lead to the same conclusions. However, one may wonder whether the analysis extends to a less restrictive sufficiency condition. We show that this is indeed the case; in fact, the following result holds:

Theorem 2 *Let* $\Omega = \mathbb{T} \times \mathbb{R}^+$*. Assume the functions* $A : \mathbb{R}^+ \to \mathbb{R}^+$, $k_0 : \mathbb{T} \to \mathbb{R}^+ \cup \{0\}$ *and* $c : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ *are continuous in their respective domains. Assume, moreover, that*

$$k_0(x) \geq \int_0^t e^{-\int_0^s A(z)\,dz} \max_{x \in \mathbb{T}} c(x,s)\,ds \quad \forall (x,t) \in \Omega\,. \tag{9}$$

Then the Problem 1 admits a unique non-negative classical solution.

Theorem 2 only requires that the consumption function is continuous and that the present discounted value of the flow of maximum consumption (on the entire space) does not exceed the initial capital stock at any point of the space. Note that the discounting is done using a (time-varying) technology parameter.

The economic interpretation of this condition is as follows. Note that (9) allows for spatial inequality in consumption; it just imposes the upper bound on the (present discounted value of the) highest values of this consumption. Moreover, given the discounting, it allows for an increasing spatial inequality in consumption over time. The upper bound imposed depends on the initial distribution of the capital stock, and in particular, the condition (9) is most stringent for the lowest initial capital stock on the circle. In other words, it is more difficult to satisfy this condition when, ceteris paribus, the initial spatial inequality in capital stock is higher.[1]

3.1 Proof of the main results

Let's start by observing that we can simplify Eq. (5) by removing the term $A(t)k(x,t)$ using the following Lemma (whose proof is presented in the Appendix):

[1] The conditions (8) and (9) hold for all positive t; hence, they imply that the integral on the right hand side converges at the limit. This implies

$$\lim_{t \to \infty} \left[e^{-\int_0^t A(s)\,ds} c(x,t) \right] = 0 \quad \forall x \in \mathbb{T}\,,$$

and

$$\lim_{t \to \infty} \left[e^{-\int_0^t A(s)\,ds} \max_{x \in \mathbb{T}} c(x,t) \right] = 0\,,$$

which constrains the consumption function to grow over time at a slightly lower rate than the time-varying technology parameter.

Lemma 1 (Equivalent solutions) *Let $k(x,t)$ and $h(x,t)$ be two positive functions defined in Ω and related to each other by*

$$h(x,t) = e^{-\int_0^t A(s)\,ds} k(x,t)\,. \tag{10}$$

Then $k(x,t)$ is a positive solution of (5) *with initial condition* (6) *if and only if $h(x,t)$ is a positive solution of*

$$h_t = h_{xx} - \gamma(x,t) \quad \forall (x,t) \in \Omega\,, \tag{11}$$

with the same initial condition (6)*, where $\gamma(x,t) = e^{-\int_0^t A(s)\,ds} c(x,t)$.*

The proofs of Theorems 1 and 2 are constructed with following steps.[2] We first find a formal solution, i.e. a Fourier series that, order by order, solves (5).

Proposition 1 (Formal solution) *Let us define, for any positive integer n, $\lambda_n = n^2$ and, for $(x,y,t) \in \mathbb{T}^2 \times \mathbb{R}^+$, the Green's function*

$$G(x,y,t) = \sum_{n \geq 0} e^{-\lambda_n t} \cos\left[n\,(x-y)\right]\,. \tag{12}$$

Then the function $h(x,t)$ given by

$$h(x,t) = \frac{1}{\pi} \int_{-\pi}^{\pi} G(x,y,t) k_0(y)\,dy - \frac{1}{\pi} \int_0^t ds \int_{-\pi}^{\pi} G(x,y,t-s)\gamma(y,s)\,dy\,, \tag{13}$$

is a formal solution of Eq. (11)*.*

The second step is to prove that the solution provided by the previous proposition is actually a classical solution.

Proposition 2 (Classical solution) *Under the above assumptions the function $G(x,y,t)$, respectively $h(x,t)$, is continuous in $\mathbb{T}^2 \times [0,+\infty)$, respectively $\mathbb{T} \times [0,+\infty)$, and twice differentiable in $\mathbb{T}^2 \times (0,+\infty)$, respectively $\mathbb{T} \times (0,+\infty)$.*

The third steps is to prove the uniqueness of the classical solution:

Proposition 3 (Uniqueness) *The classical solution of the Eq.* (11) *with initial condition $k(x,t) = k_0(x)$ is unique.*

[2] For the sake of clarity, all the proofs except those concerning the non-negativity are relegated to the Appendix.

Considering the above-mentioned results and using Lemma 1, we can conclude that

$$\begin{aligned} k(x,t) &= h(x,t)e^{\int_0^t A(s)\,ds} \qquad (14)\\ &:= e^{\int_0^t A(s)\,ds}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} G(x,y,t)k_0(y)\,dy - \frac{1}{\pi}\int_0^t ds\int_{-\pi}^{\pi} G(x,y,t-s)\gamma(y,s)\,dy\right], \end{aligned}$$

is the unique classical solution of (5) with initial condition $k_0(x)$.

The last step for proving Theorem 1 is to show that the solution given by (14) is non-negative for all $(x,t) \in \Omega$, that is the function describing the evolution of the stock of capital in space and time $k(x,t)$ is everywhere and always non-negative. For this, we need a preliminary result (Lemma 2) whose proof can be found in the Appendix.

Proposition 4 (Non-negativity) *Let $h(x,t)$ be the classical solution of the Eq.* (11) *in* Ω *with boundary condition $h(x,0) = k_0(x) \geq 0$ for all $x \in \mathbb{T}$ and assume $\gamma(x,t) \geq 0$ for all $(x,t) \in \Omega$. Under the hypotheses*

$$\gamma_{xx}(x,t) \leq 0 \quad \forall (x,t) \in \Omega\,,$$

and

$$k_0(x) \geq \int_0^t \gamma(x,s)\,ds \quad \forall (x,t) \in \Omega\,,$$

we have

$$h(x,t) \geq 0 \quad \forall (x,t) \in \Omega\,. \qquad (15)$$

Proof. Let $T > 0$, $\epsilon > 0$ and let us define the auxiliary function

$$v(x,t) = h(x,t) + \epsilon t + \int_0^t \gamma(x,s)\,ds\,,$$

for all $(x,t) \in \mathbb{T} \times [0,T]$.

A straightforward computation gives:

$$v_t - v_{xx} = h_t - h_{xx} + \epsilon + \gamma(x,t) - \int_0^t \gamma_{xx}(x,s)\,ds = \epsilon - \int_0^t \gamma_{xx}(x,s)\,ds \geq \epsilon > 0,$$

where we use the fact that $h_t - h_{xx} = -\gamma(x,t)$ and the assumption $\gamma_{xx}(x,s) \leq 0$. We can thus apply the Lemma 2 to v and conclude that it attains its minimum at some $(a,\tau) \in \mathbb{T} \times \{0\}$.

Hence, we have that for all $(x,t) \in \mathbb{T} \times [0,T]$

$$h(x,t) + \epsilon t + \int_0^t \gamma(x,s)\,ds = v(x,t) \geq v(x,0) = h(x,0) = k_0(x)\,,$$

thus

$$h(x,t) + \epsilon t \geq k_0(x) - \int_0^t \gamma(x,s)\, ds \geq 0\,,$$

where the rightmost inequality holds because of the assumption on k_0. We can finally pass to the limit $\epsilon \to 0$ and conclude that

$$h(x,t) \geq 0 \quad (x,t) \in \mathbb{T} \times [0,T]\,.$$

The arbitrariness of T completes the proof. ■

Because the function $k(x,t)$ has the same sign that $h(x,t)$, we conclude that $k(x,t)$ is also non-negative in Ω and this concludes the proof of Theorem 1.

The proof of Theorem 2 can be achieved in a very similar way; the only difference lies on the way we prove the non-negativity of the solution (14) under the assumptions of Theorem 2.

Proposition 5 (Non-negativity) *Let $h(x,t)$ be the classical solution of the Eq.* (11) *in Ω with boundary condition $h(x,0) = k_0(x) \geq 0$ for all $x \in \mathbb{T}$ and assume $\gamma(x,t) \geq 0$ for all $(x,t) \in \Omega$. Under the hypothesis*

$$k_0(x) \geq \int_0^t \max_{x\in\mathbb{T}} \gamma(x,s)\, ds \quad \forall (x,t) \in \Omega\,,$$

we have

$$h(x,t) \geq 0 \quad \forall (x,t) \in \Omega\,. \tag{16}$$

Proof. Let $T > 0$, $\epsilon > 0$ and let us define the auxiliary function

$$v(x,t) = h(x,t) + \epsilon t + \int_0^t \max_{x\in\mathbb{T}} \gamma(x,s)\, ds\,,$$

for all $(x,t) \in \mathbb{T} \times [0,T]$.

A direct computation provides $v_t - v_{xx} \geq 0$ and, thus, by Lemma 2, v attains its minimum at some $(a,\tau) \in \mathbb{T} \times \{0\}$. Then, following an argument similar to the one used in the previous Proposition we conclude that $h(x,t) + \epsilon t \geq k_0(x) - \int_0^t \max_{x\in\mathbb{T}} \gamma(x,s)\, ds \geq 0$. Passing to the limit $\epsilon \to 0$, and using the arbitrariness of T, we get the result. ■

Once again, because $k(x,t)$ and $h(x,t)$ differ by a positive function, we can conclude that $k(x,t)$ is non-negative in Ω.

Our results above show that if the initial spatial inequality in capital is not too stark and that the consumption does not differ too much across space (so as to respect (9)), a given consumption function uniquely determines the spatial growth process. However, the natural question remains whether such dynamics (always) leads to convergence of capital stock across space over time. We address this question below.

3.2 Asymptotic behavior and convergence

The aim of this sub-section is to study the asymptotic behavior of the classical solution $k(x,t)$ for large t. Our analysis relies on the behavior of the Green's function $G(x,y,t)$ for large t (that is fully characterized by Lemma 3 in the Appendix).

Let us first consider the case of time-independent technology, i.e. $A(t) = A_0$ for all t. Our main result is the following

Proposition 6 *Let (9) hold and $k(x,t)$ be the classical non-negative solution of* (5) *with initial condition $k(x,0) = k_0(x)$ and $A(t) = A_0 \in \mathbb{R}^+$. Then, assuming $\widetilde{c}(t)$ to be bounded, we have*

$$\lim_{t\to\infty} \left[k(x,t)e^{-A_0 t}\right] = \widetilde{k}_0 - \int_0^\infty e^{-A_0 s}\widetilde{c}(s)\, ds \tag{17}$$

uniformly in $\mathbb{T}$, where

$$\widetilde{k}_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} k_0(x)\, dx \quad \text{and} \quad \widetilde{c}(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} c(x,t)\, dx\,.$$

Proof. From the explicit form for the classical non-negative solution $k(x,t)$ in the case $A(t) = A_0$ we get:

$$k(x,t)e^{-A_0 t} = \frac{1}{\pi}\int_{-\pi}^{\pi} G(x,y,t)k_0(y)\, dy - \frac{1}{\pi}\int_0^t ds \int_{-\pi}^{\pi} G(x,y,t-s)e^{-A_0 s}c(y,s)\, dy\,,$$

that can be rewritten as

$$\begin{aligned} k(x,t)e^{-A_0 t} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \left[G(x,y,t)-1\right] k_0(y)\, dy + \widetilde{k}_0 \\ &- \frac{1}{\pi}\int_0^t ds \int_{-\pi}^{\pi} \left[G(x,y,t-s)-1\right] e^{-A_0 s}c(y,s)\, dy - \int_0^t e^{-A_0 s}\widetilde{c}(s)\, ds\,. \end{aligned}$$

Using Lemma 3 the first term on the top line becomes, at the limit $t \to \infty$:

$$\lim_{t\to\infty}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} \left[G(x,y,t)-1\right] k_0(y)\, dy\right] = 0\,.$$

The remaining terms can be handled as follows:

$$\begin{aligned} \left|\frac{1}{\pi}\int_0^t ds \int_{-\pi}^{\pi} \left[G(x,y,t-s)-1\right] e^{-A_0 s}c(y,s)\, dy\right| &\le \frac{1}{\pi}\int_0^t ds \int_{-\pi}^{\pi} \sum_{n\ge1} e^{-n^2 t}e^{(n^2-A_0)s}c(y,s)\, dy \\ &\le \max_{s\le t}\widetilde{c}(s)\sum_{n\ge1} e^{-n^2 t}\frac{1}{n^2-A_0}\left(e^{(n^2-A_0)t}-1\right) \\ &\le \max_{s\le t}\widetilde{c}(s)\left(e^{-A_0 t}\sum_{n\ge1}\frac{1}{n^2-A_0} - \sum_{n\ge1}\frac{e^{-n^2 t}}{n^2-A_0}\right)\,. \end{aligned}$$

The series are convergent and their sums vanish in the limit. Hence,

$$\lim_{t\to\infty}\left[\frac{1}{\pi}\int_0^t ds\int_{-\pi}^{\pi}\left[G(x,y,t-s)-1\right]e^{-A_0 s}c(y,s)\,dy\right]=0\,,$$

which concludes the proof. ■

Let us now consider the case of a time-dependent technology parameter. One can prove a similar result under the additional minor assumption that the technology parameter is always strictly positive, $A(t) > 0$.

Proposition 7 *Let condition (9) hold and $k(x,t)$ be the classical positive solution of* (5) *with initial condition $k(x,0) = k_0(x)$, $A : \mathbb{R}^+ \to \mathbb{R}^+$, and there exists $A_+ > 0$ such that $A(t) \geq A_+$ for all $t \geq 0$. Then, assuming $\widetilde{c}(t)$ to be bounded, we get*

$$\lim_{t\to\infty}\left[k(x,t)e^{-\int_0^t A(s)\,ds}\right]=\widetilde{k}_0-\int_0^{\infty}e^{-\int_0^s A(z)\,dz}\widetilde{c}(s)\,ds\,. \tag{18}$$

uniformly in $\mathbb{T}$, where

$$\widetilde{k}_0=\frac{1}{\pi}\int_{-\pi}^{\pi}k_0(x)\,dx \quad \textit{and} \quad \widetilde{c}(t)=\frac{1}{\pi}\int_{-\pi}^{\pi}c(x,t)\,dx\,.$$

Proof. The claim can be proven along the same lines of the previous Proposition, observing that we have the following bound:

$$e^{-\int_0^t A(s)\,ds}\leq e^{-A_+ t}\quad \forall\, t\geq 0\,.$$

■

We have shown that any optimal growth dynamics that satisfies the condition (9) leads to spatial convergence in capital over time. Intuitively, the capital diffusion based on spatial trade between more capital-abundant locations and their less capital-abundant neighbors, as described by the trade balance relation $\tau(x,t) = -k_{xx}(x,t)$, guarantees, for a given consumption profile that respects (9), that the initial spatial inequality in capital disappears over time. Crucially, this does not depend on a specific objective function of the social planner. This is important because setting the spatial weights in such objective function as being equal for every location is somewhat arbitrary and might not respect certain welfare criteria; our findings above imply that this equality-of-weights assumption can be relaxed.

4 Conclusion

We have shown that the asymptotic convergence in a stylized spatial AK growth model does not depend on restrictive assumptions about the objective function of the social planner. We have also generalized this finding, allowing for the time-varying technology parameter, and provided an explicit solution for the dynamics of spatial distribution of the capital stock.

Two directions for future research look particularly promising. First, the technology might depend not only on time but also differ in space, as suggested, for instance, by Quah (2002). This conjecture has been confirmed empirically. For instance, large spatial productivity differences have been documented by Acemoglu and Dell (2010) for Latin America, which also show that within-country differences are much larger than the between-country ones, and suggest that these differences are shaped by institutional features (in particular, by the distribution of political power locally). An alternative explanation is that agglomeration externalities make firms more productive, as has been shown by Combes et al. (2012). This probably has to do with the space-varying innovation incentives of firms, as discussed and modelled by Desmet and Rossi-Hansberg (2012, 2014). This calls for an analysis of the growth dynamics with space- and time-varying technological parameter, along the lines of this paper.[3]

Second, the spatial dynamics of the capital stock might not always be described appropriately by a diffusion process. Empirically, Desmet and Rossi-Hansberg (2009) show that diffusive dynamics applies well to established sectors (such as manufacturing in 1970-2000s); however, for younger sectors (e.g., manufacturing at the beginning of the 20th century or retail and financial sectors in 1970-2000s), the dynamics is (at least locally) agglomerative. This requires a modelling of the equation of motion of capital richer than the one presented in this paper. Ideally, such a model would also capture the transition process from a non-diffusive to a diffusive dynamics.

A Technical proofs

Lemma 1 *Let $k(x,t)$ and $h(x,t)$ be two positive functions defined in Ω and related to each other by*

$$h(x,t) = e^{-\int_0^t A(s)\,ds} k(x,t)\,. \tag{19}$$

[3]Notably, Comin et al. (2013) document the fundamental importance of the geographic distance for technology diffusion, which shows why an explicit modelling of the spatial dimension is crucial.

Then $k(x,t)$ is a positive solution of (5) *with initial condition* (6) *if and only if $h(x,t)$ is a positive solution of*

$$h_t = h_{xx} - \gamma(x,t) \quad \forall (x,t) \in \Omega\,, \tag{20}$$

where $\gamma(x,t) = e^{-\int_0^t A(s)\,ds} c(x,t)$, with the initial condition (6)*.*

Proof. The proof is obtained by direct computation. In fact, for all $(x,t) \in \Omega$ we get:

$$\begin{aligned} h_t - h_{xx} &= -A(t)h(x,t) + e^{-\int_0^t A(s)\,ds} k_t(x,t) - e^{-\int_0^t A(s)\,ds} k_{xx}(x,t) \\ &= -A(t)h(x,t) + e^{-\int_0^t A(s)\,ds} \left[A(t)k(x,t) - c(x,t)\right] = -e^{-\int_0^t A(s)\,ds} c(x,t) = -\gamma(x,t)\,. \end{aligned}$$

Finally

$$h(x,0) = k(x,0) = k_0(x)\,.$$

■

Proposition 1 (Formal solution). *Let us define, for any positive integer n, $\lambda_n = n^2$ and let us define for $(x,y,t) \in T^2 \times R^+$, the Green's function*

$$G(x,y,t) = \sum_{n \geq 0} e^{-\lambda_n t} \cos\left[n\,(x-y)\right]\,. \tag{21}$$

Then the function $h(x,t)$ given by

$$h(x,t) = \frac{1}{\pi} \int_{-\pi}^{\pi} G(x,y,t) k_0(y)\,dy - \frac{1}{\pi} \int_0^t ds \int_{-\pi}^{\pi} G(x,y,t-s)\gamma(y,s)\,dy\,, \tag{22}$$

is a formal solution of Eq. (20)*.*

Proof. Let us observe that formal Green's function satisfies for all $t > 0$ and $(x,y) \in \mathbb{T}^2$ the following homogeneous PDE

$$\partial_t G(x,y,t) = \partial_x^2 G(x,y,t)\,, \tag{23}$$

and moreover,

$$\lim_{t \to 0} G(x,y,t) = \delta(x-y)\,, \tag{24}$$

where $\delta(x-y)$ is the Dirac delta-function.

Then, to prove our claim, it is sufficient to plug $h(x,t)$ given by (22) into Eq. (20) and by exchanging the derivatives with the integrals one easily finds:

$$\begin{aligned} h_t - h_{xx} &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\partial_t G(x,y,t) - \partial_x^2 G(x,y,t)\right] k_0(y)\,dy \\ &- \frac{1}{\pi} \int_0^t ds \int_{-\pi}^{\pi} \left[\partial_t G(x,y,t-s) - \partial_x^2 G(x,y,t-s)\right] \gamma(y,s)\,dy\,, \\ &- \lim_{t \to s} \frac{1}{\pi} \int_{-\pi}^{\pi} G(x,y,t-s)\gamma(y,s)\,dy\,, \end{aligned}$$

Using Eq. (23) the first two terms on the right-hand side vanish, while using (24) the remaining term reduces to $\gamma(x,t)$. We can thus conclude that:

$$h_t - h_{xx} = -\gamma(x,t)\,,$$

hence $h(x,t)$ solves (20).

To show that $h(x,t)$ satisfies the initial condition (6), it is enough to pass to the limit $t \to 0$ in the definition (22) and to make use of (24). ■

Proposition 2 (Classical solution). *Under the above assumptions, the function $G(x,y,t)$, respectively $h(x,t)$, is continuous in $T^2 \times [0,+\infty)$, respectively $T \times [0,+\infty)$, and twice differentiable in $T^2 \times (0,+\infty)$, respectively $T \times (0,+\infty)$.*

Proof. The claim is proven by showing that $G(x,y,t)$ is the uniform limit of smooth functions on Ω, and thus it is itself a smooth function.

We start by rewriting (21) as:

$$G(x,y,t) = 1 + \sum_{n\geq 1} e^{-\lambda_n t} \cos\left[n\,(x-y)\right]\,,$$

hence, after observing that for all $t > 0$ and $(x,y) \in \mathbb{T}^2$, one has:

$$|e^{-\lambda_n t} \cos\left[n\,(x-y)\right]| \leq e^{-n^2 t}\,,$$

and we can conclude that

$$\Big|\sum_{n\geq 1} e^{-\lambda_n t} \cos\left[n\,(x-y)\right]\Big| \leq e^{-t} \sum_{m\geq 0} e^{-2mt} = \frac{e^{-t}}{1-e^{-2t}}\,,$$

which proves the norm convergence of the sum. This, in turn, implies the uniform convergence and thus the smoothness of $G(x,y,t)$.

The claim for $h(x,t)$ follows easily from its definition and is thus skipped. ■

Proposition 3 (Uniqueness). *The classical solution of the Eq.* (20) *with initial condition $k(x,t) = k_0(x)$ is unique.*

Proof. Let us assume, on the contrary, the existence of two distinct classical solutions $h_1(x,t)$ and $h_2(x,t)$ of the Eq. (20) with initial conditions $h_i(x,0) = k_0(x)$ for $i = 1,2$. Then, the function $f(x,t) = h_1(x,t) - h_2(x,t)$ solves the equation

$$f_t = f_{xx} \quad \forall (x,t) \in \Omega\,,$$

with initial condition $f(x,0) = 0$. However, then, using the Green's function $G(x,y,t)$ defined above, one easily concludes that $f(x,t)$ identically vanishes on Ω and thus $h_1 \equiv h_2$.

Lemma 2 *Let $T > 0$ and let v be a smooth function satisfying the inequality*

$$v_t - v_{xx} > 0 \quad \forall (x,t) \in \mathbb{T} \times [0,T] \, .$$

Then, v attains its minimum at some $(a,\tau) \in \mathbb{T} \times \{0\}$.

Proof. Because of the smoothness assumption, v must attain its minimum somewhere on the compact set $\mathbb{T} \times [0,T]$. Let's assume, by contradiction, the minimum (a,τ) to lie in $\mathbb{T} \times (0,T]$. Then, by elementary calculus we have [4]

$$v_t(a,\tau) \leq 0 \quad \text{and} \quad v_{xx}(a,\tau) \geq 0 \, ,$$

from which we straightforwardly calculate:

$$v_t(a,\tau) - v_{xx}(a,\tau) \leq 0 \, ,$$

contradicting the hypothesis. Hence, we must have $(a,\tau) \in \mathbb{T} \times \{0\}$. ■

Lemma 3 *Let $G(x,y,t)$ be the Green's function defined by (21); then*

$$\lim_{t\to\infty} |G(x,y,t) - 1| = 0 \, , \tag{25}$$

uniformly for $(x,y) \in \mathbb{T}^2$.

Proof. Using the definition, we can write

$$G(x,y,t) - 1 = \sum_{n\geq 1} e^{-\lambda_n t} \cos\left[n\,(x-y)\right] \, ,$$

from which we get

$$|G(x,y,t) - 1| \leq \sum_{n\geq 1} e^{-n^2 t} \leq \frac{e^{-t}}{1-e^{-2t}} \, .$$

Hence Eq. (25) follows directly. ■

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