Weak Ex Ante Collusion and Design of Supervisory Institutions[1]

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Abstract

A Principal seeks to design a mechanism for an agent (privately informed regarding production cost with a continuous distribution) and a supervisor/intermediary (with a noisy signal of the agent's cost) that collude *ex ante*, i.e., on *both* participation and reporting decisions. Collusion is 'weak' in the sense that neither colluding party can commit to how they would behave if they fail to mutually agree to a side-contract. We provide conditions under which the Principal's problem reduces to selecting weak collusion-proof (WCP) allocations. We characterize WCP allocations, and use this to show that it is always valuable to employ the supervisor. Delegation is optimal, but only if supplemented by an appeal/dispute settlement mechanism mediated by the Principal, which serves as an outside option for coalitional bargaining. Changes in bargaining power within the coalition have no effect, while altruism of the supervisor towards the agent makes the Principal worse off.

KEYWORDS: mechanism design, intermediation, supervision, collusion, delegation

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1 Introduction

The potential for collusion is widely acknowledged to be a serious problem for a Principal who relies on information provided by an expert intermediary or supervisor to design a contract for an agent. Examples of such contexts abound: an investor that relies on an investment bank or rating agency for information necessary to decide on financing an entrepreneur; shareholders that rely on outside directors of a company to supervise its CEO; an owner or CEO that relies on a product manager for information needed to set production targets and compensation for workers or suppliers; or a government that relies on a regulator to advise on rates for a public utility. In these settings the supervisor is typically better informed about the agent's productivity or cost than the Principal, but less informed than the agent. Eliciting the supervisor's information becomes problematic when he is willing to misreport information in exchange for suitable side payments with the agent, which cannot be observed by the Principal.

The severity of the collusion problem depends sensitively on precise institutional details. Early literature on the mechanism design problem with collusion (e.g., Tirole (1986), Laffont and Tirole (1993)) was based on the assumption of hard information (where the supervisor cannot lie, and can only withhold information), and exogenous transaction costs of collusion. Subsequent literature has considered contexts where the collusion problem is harder to control, owing to soft information (which allows the supervisor to report anything) and absence of exogenous transaction costs of collusion. A large part of the literature considers only the possibility of *interim* collusion, where supervisor and agent can collude over reporting decisions, but not whether to participate in the mechanism (e.g., Laffont and Martimort (1997, 2000), Faure-Grimaud, Laffont and Martimort (2003), Che and Kim (2006), Celik (2009)). In the context of auctions or team production, a number of authors have studied the consequences of *ex ante* collusion where agents collude over participation decisions as well (e.g., Mookherjee and Tsumagari (2004), Dequiedt (2007), Pavlov (2008) and

Che and Kim (2009)). How this affects the design of mechanisms in contexts of hierarchical supervision has however not been previously studied in the literature. Given the critical role of participation decisions in determining the magnitude of information rents, *ex ante* collusion obviously has more severe consequences for what the Principal can achieve. Its importance is undeniable in contexts where colluding agents have pre-existing relationships and communication opportunities with one another, before they are approached by the Principal.

This paper studies consequences of a version of *ex ante* collusion for design of hierarchical supervision mechanisms. We consider a setting where an agent produces a divisible good at a constant unit cost whose realization is known to him privately, and the Principal and the supervisor have a prior over this cost which is continuously distributed over some interval. The supervisor costlessly updates this prior on the basis of a noisy signal of the agent's cost. The signal is only partially informative: it can take a finite number of possible realizations. The agent also observes the realization of this signal. The supervisor and agent can enter into a side-contract which coordinates on respective participation and cost/signal reports to the Principal, as well as a private side payment, conditioned on a private cost report made by the agent to the supervisor. The side contract is designed and offered by the supervisor to the agent, though we subsequently show that our results extend to contexts where they are designed instead by a third party that maximizes a weighted sum of their interim payoffs. The side contract and the internal communication and transfers within the coalition are unobserved by the Principal. We study a specific version of this problem, referred to as *weak ex ante* collusion, where neither colluding party can commit to how they will behave in the event that they fail to agree on a side-contract.[5]

The solution concept we employ requires that an allocation designed by the Prin-

[5]The literature on collusion in auctions has considered both (strong and weak) forms of ex ante collusion, where colluding parties respectively can and cannot make such commitments. We postpone the study of strong ex ante collusion to a future paper.

cipal should (besides meeting interim participation constraints) leave no room for design of a non-null side contract (and selection of a weak Perfect Bayesian Equilibrium of the resulting continuation game) which is Pareto improving for the colluding parties, and generates a strict improvement for the designer of the side contract.[6]

Our principal results are the following. These pertain to a context where the Principal's benefit function is strictly concave and satisfies Inada conditions, so optimal allocations are always interior. We mention how they are modified in the case of a linear benefit function with a capacity constraint.

(a) Delegation to the supervisor (DS), where the Principal contracts only with the supervisor and delegates to her the authority to contract with the agent, is strictly dominated by not appointing any supervisor (NS).[7] Hence delegation cannot be rationalized as an optimal response of the Principal to weak *ex ante* collusion. This can be contrasted to the optimality of delegation with *interim* collusion, when there are two possible types of the agent and two possible signals of the supervisor (Faure-Grimaud, Laffont and Martimort (2003)).[8]

[6] The Appendix shows that the Principal would not benefit from allowing collusion to occur on the equilibrium path. Moreover, in the formulation of the side contracting problem there is no loss of generality in restricting attention to side contracts that are always accepted by the colluding parties, thereby addressing a problem highlighted by Celik and Peters (2011). It is also shown that WCP allocations can alternatively be given a purely 'noncooperative' justification by imposing restrictions on off-equilibrium-path beliefs, which generalize the traditional notion of 'passive beliefs' employed in previous papers (e.g., Laffont and Martimort (1997, 2000), Faure-Grimaud, Laffont and Martimort (2003)), in a manner that addresses the Celik-Peters problem. This restriction essentially amounts to requiring that beliefs be independent of the side contract offered, or whether or not it is offered.

[7] If side contracts are designed by a third party that maximizes a weighted sum of the supervisor's and agent's payoffs, the same result applies for delegation to the third party, as long as the third party assigns a positive welfare weight to the supervisor's payoff. When the supervisor is assigned a zero welfare weight, DS turns out to be equivalent to NS.

[8] An important reason for the difference in results is that ex ante collusion makes it much harder

(b) Centralized contracting with the supervisor and agent (CS) strictly dominates NS, so it is valuable for the Principal to employ the supervisor and induce full revelation of information, despite *ex ante* collusion.

(c) Sufficient conditions are provided for collusion to be costly for the Principal: the support of the conditional cost distribution is independent of the supervisor's signal, conditional distributions satisfy suitable monotonicity and monotone likelihood properties, and the Principal's gross benefit function exhibits sufficient curvature. However, in contexts where the Principal's benefit function is linear (analogous to the context of an auction), there are cases where the second-best can be achieved (and also others where it cannot).

(d) Any allocation that is implementable with weak collusion can be implemented by a modified form of delegation, in which the Principal communicates and transacts with only the supervisor *on the equilibrium path.*[9] The mechanism leaves open the room for the agent to trigger a switch to a centralized mechanism (the grand contract) where both agent and supervisor make independent reports to the Principal. This may be thought of as an 'appeals' or 'dispute settlement'

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for the Principal to extract rents from the supervisor. In fact, it turns out that with weak ex ante collusion in the two-type-two-signal case, the Principal never benefits from appointing a supervisor (this result is not provided in this paper, and is available on request). As subsequently explained in more detail, ex ante collusion effectively allows the supervisor to postpone participation decisions until after learning the agent's true type. Delegation is then associated with the classic problem of 'double marginalization of rents' wherein both the agent and the supervisor earn information rents. It is thereby inferior to not hiring a supervisor at all, where the rents of the supervisor can be eliminated.

[9] This corresponds to a hierarchical delegation arrangement where the Principal asks the supervisor to initially communicate and transact with the agent, and then submit a joint participation decision and cost-cum-signal report to the Principal on behalf of the coalition. These reports determine an output target and aggregate payment for the coalition made by the Principal to the supervisor, who subsequently relays the output target and makes a corresponding out-of-pocket payment to the agent.

procedure mediated by the Principal, which is not activated in equilibrium but plays a key role by determining outside options for coalition partners when they negotiate a side-contract. The reverse pattern of delegation — where the Principal communicates only with the agent on the equilibrium path, while reserving the right to consult the supervisor depending on the agent's reports — is also capable of implementing the optimal WCP allocation.

(e) Given the outside options determined by the grand contract set by the Principal, the allocation of bargaining power (i.e., allocation of welfare weights) within the coalition does not affect the set of implementable allocations with weak collusion.[10] Optimal mechanisms are no different if the agent makes a take-it-or-leave-it offer of a side contract to the supervisor, or if there is a third-party that mediates the collusion. This is an implication of weak collusion, where outside options are independent of bargaining power.[11]

(f) Appointing a supervisor exhibiting some altruism with respect to the agent, or an increase in the degree of such altruism, makes the Principal worse off.

These results offer interesting implications for organizational design in varied settings. Our theory rationalizes the widespread prevalence of supervisors, despite the potential for collusion. Moreover, collusion is typically costly for the Principal, including those where interim collusion can be overcome via mechanisms of the sort constructed by Motta (2009). Our theory does not rationalize unconditional delegation of authority to the supervisor; instead, delegation needs to be supplemented by scope for agents to 'appeal' and trigger direct communications with the Principal.

[10]An analogous result for the case of *interim* collusion is obtained by Faure-Grimaud, Laffont and Martimort (2003).

[11]In strong collusion (Quesada (2004), Dequiedt (2007), Che and Kim (2009)) where the side contract designer can commit to playing the subsequent grand contract in suitable ways, outside options depend on the allocation of bargaining power, which thereby affects the set of implementable allocations.

Such appeals do not arise in equilibrium. But the scope for such appeals indirectly promote the agent's bargaining power with the supervisor (by altering outside options in coalitional bargaining), which reduces the severity of the double-marginalization-of-rents (DMR) problem and thus ends up benefitting the Principal. Within firms, it explains the role of worker rights to appeal the evaluations reported by their managers to higher level managers or an ombudsman appointed for this purpose. This echoes Williamson's (1975) view of such dispute settlement procedures as an advantage of hierarchies over market relationships. It is also similar to Hirschman's (1970) view of organizations as including exit and voice options, in contrast to market relationships which involve only exit.

Result (e) states that with weak collusion, direct changes in bargaining power (represented by welfare weights in coalitional bargaining) make no difference. This has implications for the way that supervisors and agents are matched, e.g., whether an agent should be allowed to select an auditor on a competitive market, or whether the Principal should appoint the auditor instead. This result is however likely to be sensitive to the collusion concept which does not allow either colluding partner to make commitments regarding how it will report to the Principal should collusion negotiations break down. When such commitments are possible, the notion of weak collusion is not suitable, and should be replaced by a suitable notion of 'strong' collusion. We hope to explore this extension in future research.

Result (f) above implies that the Principal ought to appoint 'outside' self-interested supervisors rather than 'insiders' likely to be altruistic towards the agent. In the context of corporate governance, for instance, this is an argument in favor of appointing 'outsiders' rather than 'insiders' to a company's Board of Directors.[12] In the context of regulation, it confirms the normal intuition in favor of preventing any direct conflict of interest for the supervisor (e.g., who should not have a financial stake in the

[12] See Harris and Raviv (2008) for a model based on incomplete contracts where this result may not hold in some settings.

agent's fortunes, nor have any social or personal connections with the agent). This result is not entirely obvious as altruism has some benefits for the Principal: it limits the inclination of the supervisor to extract rents from the agent that is the source of the DMR problem.

The paper is organized as follows. Section 2 describes relation to existing literature in more detail. Section 3 introduces the model, followed by definition and characterization of WCP allocations. The main results concerning properties of optimal weak-collusion-proof mechanisms are presented in Section 4 for the polar model, where optimal allocations are always interior and the supervisor has all the bargaining power within the coalition. Section 5 then considers a number of extensions: where (a) the Principal can implement the optimal WCP allocation by a modified delegation arrangement; (b) alternative allocations of bargaining power within the coalition, wherein side contracts are designed and offered by a third party maximizing a weighted sum of supervisor and agent's payoffs; (c) the supervisor may exhibit altruism towards the agent, and (d) the Principal's gross benefit function is linear (whereby optimal allocations are never interior). Finally, Section 6 concludes with a summary and directions for future work.

2 Relation to Existing Literature

The literature on mechanism design with collusion can be classified by the context (auctions, team production or supervision), the nature of collusion (ex ante or interim, weak or strong collusion), and whether type spaces are discrete or continuous. Auctions and team production involve multiple privately informed agents and no supervisor. For auctions, Dequiedt (2007) considers strong ex ante collusion with binary agent types and shows that efficient collusion is possible, implying that the second-best cannot be implemented. In contrast, Pavlov (2008) considers a model with continuous types where the second-best can be implemented with weak ex ante collusion, and Che and Kim (2009) find the same result with either weak or strong ex ante collusion with continuous types.

Team production with binary types is studied by Laffont and Martimort (1997), who show the second best can be implemented with weak interim collusion; this analysis is extended to a public goods context in Laffont and Martimort (2000) where the role of correlation of types is explored. Che and Kim (2006) show how second-best allocations can be implemented in a team production context with continuous types in the presence of weak interim collusion. Quesada (2004) on the other hand shows strong ex ante collusion is costly in a team production model with binary types. Mookherjee and Tsumagari (2004) show delegation to one of the agents is worse than centralized contracting in the presence of weak ex ante collusion. The logic of this result is similar to that underlying our result that delegation to the supervisor is worse than not appointing a supervisor. Their paper also considers delegation to a supervisor who is perfectly informed about the costs of each agent, and show that its value relative to centralized contracting depends on complementarity or substitutability between inputs supplied by different agents. The current paper differs insofar as there is only one agent, and there is asymmetric information within the supervisor-agent coalition owing to the supervisor receiving a noisy signal of the agent's cost. This friction in coalitional bargaining plays a key role in the current paper.

In the context of collusion between a supervisor and agent, existing models (with the exception of Mookherjee-Tsumagari (2004)) have explored interim collusion only. Faure-Grimaud, Laffont and Martimort (2003) consider a model with binary types and signals (with full support for conditional distributions), a risk-averse supervisor where collusion is costly, where (unconditional) delegation turns out to be an optimal response to collusion. Celik (2009) considers a model with three types and two signals (where the support of conditional distributions depends on the signal), and risk neutral supervisor and agent, in which unconditional delegation is dominated by no supervision, which in turn is dominated strictly by centralized contracting with supervision. Celik's results are similar to ours, but he considers interim rather than ex ante collusion. Our results can be viewed as finding that the results he derived in the context of interim collusion with a special information structure happen to obtain quite generally with ex ante collusion and continuous types. The need to examine ex ante rather than interim collusion is highlighted by Motta (2009) who shows that collusion can be rendered costless in models with discrete type and signal spaces and interim collusion, by using mechanisms where the Principal offers a menu of contracts to the agent which the latter must respond to before colluding with the supervisor.

3 Model

3.1 Environment

We consider an organization composed of a principal (P), an agent (A) and a supervisor (S). P can hire A who delivers an output $q \geq 0$ at a personal cost of θq. P's return from q is $V(q)$ where $V(q)$ is twice continuously differentiable, increasing and strictly concave satisfying $\lim_{q \to 0} V^{'}(q) = +\infty$, $\lim_{q \to +\infty} V^{'}(q) = 0$ and $V(0) = 0$. These conditions imply that $q^*(\theta) \equiv \arg_q \max V(q) - \theta q$ is continuously differentiable, positive on $\theta \in [0, \infty)$ and strictly decreasing. In Section 5.4 we shall describe how the results are modified when V is linear and subject to a capacity constraint.

We use θ to denote a random variable whose realization is privately observed by A. It is common knowledge that everybody shares a common prior $F(\theta)$ over θ on the interval $\Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \Re_+$. F has a density function $f(\theta)$ which is continuously differentiable and everywhere positive on its support. The 'virtual cost' $H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$ is assumed to be strictly increasing in θ.

The supervisor S plays no role in production, and costlessly acquires an informative signal η about A's cost θ. The set of possible realizations of η is Π, a finite set with $\#\Pi \geq 2$. It is common knowledge that the realization of η is observed by both

S and A. $a(\eta \mid \theta) \in [0,1]$ denotes the likelihood function of η conditional on θ, which is common knowledge among all agents. $a(\eta \mid \theta)$ is continuously differentiable and positive on $\Theta(\eta)$, where $\Theta(\eta)$ denotes the set of values of θ for which signal η can arise with positive probability. We assume $\Theta(\eta)$ is an interval, for every $\eta \in \Pi$. Define $\underline{\theta}(\eta) \equiv \inf \Theta(\eta)$ and $\bar{\theta}(\eta) \equiv \sup \Theta(\eta)$. We assume that for any $\eta \in \Pi$, $a(\eta \mid \theta)$ is not a constant function on Θ, and there are some portions of θ with positive measure such that $a(\eta \mid \theta) \neq a(\eta^{'} \mid \theta)$ for any $\eta, \eta^{'} \in \Pi$. In this sense each possible signal realization conveys information about the agent's cost. The information conveyed is partial, since Π is finite.

The conditional density function and the conditional distribution function are respectively denoted by $f(\theta \mid \eta) \equiv f(\theta)a(\eta \mid \theta)/p(\eta)$ (where $p(\eta) \equiv \int_{\underline{\theta}(\eta)}^{\bar{\theta}(\eta)} f(\tilde{\theta})a(\eta \mid \tilde{\theta})d\tilde{\theta})$ and $F(\theta \mid \eta) \equiv \int_{\underline{\theta}(\eta)}^{\theta} f(\tilde{\theta} \mid \eta)d\tilde{\theta}$. The 'virtual' cost conditional on the signal η is $h(\theta \mid \eta) \equiv \theta + \frac{F(\theta|\eta)}{f(\theta|\eta)}$. We do not impose any monotonicity assumption for $h(\theta \mid \eta)$. Let $\hat{h}(\theta \mid \eta)$ be constructed from $h(\theta \mid \eta)$ and $F(\theta \mid \eta)$ by the ironing procedure introduced by Myerson (1981).

All players are risk neutral. P's objective is to maximize the expected value of $V(q)$, less expected payment to A and S, represented by X_A and X_S respectively. S's objective is to maximize expected transfers $X_S - t$ where t is a transfer from S to A. A seeks to maximize expected transfers received, less expected production costs, $X_A + t - \theta q$. Both A and S have outside options equal to 0.

In this environment, a feasible (deterministic) allocation is represented by $(u_A, u_S, q) = \{(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta)) \in \Re^2 \times \Re_+ \mid (\theta,\eta) \in K\}$ where $K \equiv \{(\theta,\eta) \mid \eta \in \Pi, \theta \in \Theta(\eta)\}$, u_S, u_A denotes S and A's payoff respectively, and q represents the production level. P's payoff equals $u_P = V(q) - u_S - u_A - \theta q$. These payoffs relate to transfers and productions as follows: $u_A \equiv X_A + t - \theta q; u_S \equiv X_S - t; u_P \equiv V(q) - X_S - X_A$.

3.2 Mechanism in the Absence of Collusion

Consider as a benchmark the case where A and S do not collude, and P designs contracts for both. We call this organization NC (no collusion). Owing to risk-neutrality of all parties and concavity of V, P can restrict attention to a deterministic grand contract:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S); M_A, M_S)$$

where M_A (resp. M_S) is a message set for A (resp. S). This mechanism assigns a deterministic allocation, i.e. transfers X_S, X_A and output q, for any message $(m_A, m_S) \in M_A \times M_S$. M_A includes A's exit option $e_A \in M_A$, with the property that $m_A = e_A$ implies $X_A = q = 0$ for any $m_S \in M_S$. Similarly M_S includes S's exit option $e_S \in M_S$, where $m_S = e_S$ implies $X_S = 0$ for any $m_A \in M_A$. The set of all possible deterministic grand contracts is denoted by $\mathcal{GC}$.

A grand contract induces a Bayesian game of incomplete information between A and S. Let $p(\eta)$ denote a set of beliefs held by S regarding the distribution of θ, in states where signal η has been realized. The posterior beliefs of S based on Bayesian updating of prior beliefs on the basis of observation of η alone are denoted by $p_\emptyset(\eta)$.

Definition 1 *A Bayesian equilibrium of the game played by A and S in state η relative to beliefs $p(\eta)$ is a set of functions $c \equiv (m_A(\theta, \eta); m_S(\eta))$ (where m_A maps K into M_A, while m_S maps Π into M_S) such that the following conditions are satisfied for all $\theta \in [\underline{\theta}(\eta), \bar{\theta}(\eta)]$:*

$$m_A(\theta, \eta) \in \arg\max_{m_A \in M_A}[X_A(m_A, m_S(\eta)) - \theta q(m_A, m_S(\eta))] \tag{1}$$

$$m_S(\eta) \in \arg\max_{m_S \in M_S} E_{p(\eta)}[X_S(m_A(\theta, \eta), m_S)] \tag{2}$$

where $E_{p(\eta)}$ denotes expectation taken with respect to beliefs $p(\eta)$. $C(p(\eta); \eta)$ denotes the set of Bayesian equilibria corresponding to the beliefs $p(\eta)$ in state η.

The timing of events in NC is as follows.

(NC1) A observes θ and η, S observes η.

(NC2) P offers the grand contract $GC \in \mathcal{GC}$, and for any $\eta \in \Pi$ recommends a Bayesian equilibrium $c(p_{\emptyset}(\eta);\eta)$ relative to posterior beliefs $p_{\emptyset}(\eta)$ based on Bayesian updating by S on the basis of observation of η alone.

(NC3) A and S play the recommended Bayesian equilibrium.

The order of the timing between (NC1) and (NC2) can be interchanged without altering any of the results. If P offers a null contract to S (defined by the property that M_S is the empty set and $X_S = 0$), this is an organization without a supervisor, which we will denote by NS. Such an organization obviously leaves no scope for collusion between A and S.

It is well-known that in NC the Principal can restrict attention to direct revelation games, where M_A and M_S reduce to reports of private information, besides participation decisions. Define the *second-best allocation* $(u_A^{SB}, u_S^{SB}, q^{SB})$ as follows:

$$u_A^{SB}(\theta,\eta) = \int_{\theta}^{\bar{\theta}(\eta)} q^{SB}(y,\eta)dy,$$

$$E[u_S^{SB}(\theta,\eta) \mid \eta] = 0$$

and

$$q^{SB}(\theta,\eta) \equiv q^*(\hat{h}(\theta \mid \eta)) = \arg\max_q [V(q) - \hat{h}(\theta \mid \eta)q]$$

where $\hat{h}(\theta \mid \eta)$ is constructed from $h(\theta \mid \eta)$ and $F(\theta \mid \eta)$ by the ironing procedure. It is well-known that this is the optimal allocation in NC, where P observes η directly. It turns out that in NC it is possible for the second-best to be implemented as a unique Bayesian equilibrium.[13]

3.3 Mechanism with Weak Ex Ante Collusion

Now we describe the game played with *weak ex ante* collusion. The 'ex ante' feature refers to the assumption that collusion takes the form of communication and side-

[13]A proof is available on request.

contracting between A and S, which takes place before they respond to P's offer of the grand contract (including participation decisions). This is distinguished from (interim) collusion where they do not collude on their participation decisions, but collude on the reports they send to P and enter into side payments in the event of joint participation. The 'weak' adjective additionally refers to the lack of commitment power of either colluding partner with respect to how they would behave (i.e., play the grand contract) in the event that they fail to agree on the side contract. In this event they would play the grand contract noncooperatively (relative to beliefs formed subsequent to the breakdown of the side contract).

The game with weak ex ante collusion is different from the game without collusion following stage NC2. At that point, A and S can enter into a side-contract in which A sends a message to S following which they jointly decide on participation, reporting and side-payments. The side-contract is unobserved by P. As in existing literature, we assume the side-contract is costlessly enforceable. Moreover we assume S has all the bargaining power *vis-a-vis* A: S can make a take-it-or-leave-it offer of a side-contract. This assumption turns out to be inessential: Section 5.2 explains how the same results obtain with side contracts offered by an uninformed third party that assigns arbitrary welfare weights to the supervisor and agent. After S offers the side contract, A retains the option of rejecting it; given that A's true cost is not known to S, this still enables A to earn some rents. This information friction within the coalition plays a key role in our analysis.

The game replaces (NC3) above (while (NC1) and (NC2) are unchanged) by the following three-stage subgame (conditional on any $\eta \in \Pi$):

(i) S offers a side-contract SC which determines for any $\tilde{\theta} \in \Theta(\eta)$ to be privately reported by A to S, a probability distribution over joint messages $(m_A, m_S) \in M_A \times M_S$, and a side payment from S to A.[14] Formally, it is a pair of functions

[14] The option of randomizing over possible messages is useful for technical reasons. Owing to quasilinearity of payoffs, there is no need to randomize over side transfers.

$\{\tilde{m}(\tilde{\theta}, \eta), t(\tilde{\theta}, \eta)\}$ where $\tilde{m}(\theta, \eta) : \Theta(\eta) \times \{\eta\} \longrightarrow \Delta(M_A \times M_S)$, the set of probability measures over $M_A \times M_S$, and $t : \Theta(\eta) \times \{\eta\} \longrightarrow \Re$. The case where S does not offer a side contract is represented by a null side-contract (NSC) with zero side payments ($t(\theta, \eta) \equiv 0$), and (deterministic) messages $(m_A(\theta, \eta); m_S(\eta))$ the same as those in the Bayesian equilibrium of the grand contract recommended by the Principal. We abuse terminology slightly and refer to the situation where no side contract is offered as one where NSC is offered.

(ii) A either accepts or rejects the SC offered, and the game continues as follows.

(iii) If A accepts the offered SC, he sends a private report $\theta' \in \Theta(\eta)$ to S, following which the SC is executed. If A rejects SC, S updates his beliefs to $p(SC; \eta)$ which is restricted to be $p_{\emptyset}(\eta)$ if NSC was offered in stage (i) above.[15] A and S then play a Bayesian equilibrium c of the grand contract relative to beliefs $p(SC; \eta)$.

We now introduce the notion of weak collusion proofness. A justification for this solution concept is provided in Section 3.5 below.

Informally, an allocation is weakly collusion proof if the supervisor cannot benefit from offering a non-null side contract when the Principal selects a grand contract based on the associated direct revelation mechanism (i.e., when agent and supervisor make consistent reports about the state, the allocation corresponding to that state is chosen). This requires the null side contract to be the optimal side contract for S, when the outside option of A corresponds to his payoff resulting from the allocation.

Before proceeding to the formal definition, note that a *deterministic allocation* can be represented by payoff functions $(u_A(\theta, \eta), u_S(\theta, \eta))$ of the true state (θ, η) combined with the output function $q(\theta, \eta)$, as these determine the Principal's pay-

[15]This ensures that it is immaterial whether or not NSC was accepted or rejected, since in either case they play the grand contract non-cooperatively with prior beliefs.

off function $u_P(\theta, \eta) \equiv V(q(\theta, \eta)) - u_S(\theta, \eta) - u_A(\theta, \eta) - \theta q(\theta, \eta)$, and the aggregate net transfers of S (equals $u_S(\theta, \eta)$) and A (equals $u_A(\theta, \eta) + \theta q(\theta, \eta)$). For technical convenience we consider randomized allocations, though it will turn out they will never actually need to be used on the equilibrium path. In a randomized allocation, $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta))$ denotes the expected payoffs of A, S and the expected output, conditional on the state (θ, η). For (conditional expected) allocation $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta))$, define functions $(\hat{X}(m), \hat{q}(m))$ on domain $m \in \hat{M} \equiv K \cup \{e\}$ (where $K \equiv \{(\theta, \eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$) as follows:

$$(\hat{X}(\theta, \eta), \hat{q}(\theta, \eta)) = (u_A(\theta, \eta) + \theta q(\theta, \eta) + u_S(\theta, \eta), q(\theta, \eta))$$

$$(\hat{X}(e), \hat{q}(e)) = (0, 0)$$

$(\hat{X}(\theta, \eta), \hat{q}(\theta, \eta))$ denote corresponding expected values of the sum of payments $X_S + X_A$ made by the principal, and the output delivered, in state (θ, η). Also, let $\Delta(\hat{M})$ denote the set of the probability measures on $\hat{M}$, and use $\tilde{m} \in \Delta(\hat{M})$ to denote a randomized message submitted by the coalition to P. With a slight abuse of notation, we shall denote the corresponding conditional expected allocation by $(\hat{X}(\tilde{m}), \hat{q}(\tilde{m}))$, which is defined on $\Delta(\hat{M})$. $\tilde{m} = (\theta, \eta)$ or e will be used to denote the probability measure concentrated at (θ, η) or e respectively.

S's choice of an optimal (randomized) side-contract can be formally posed as follows. Given a grand contract and a noncooperative equilibrium recommended by P, let the corresponding conditional expected allocation as defined above be denoted by $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta))$ and $(\hat{X}(\tilde{m}), \hat{q}(\tilde{m}))$. For any $\eta \in \Pi$, the associated side-contracting problem $P(\eta)$ is to select $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta, \eta))$ to maximize S's expected payoff

$$E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta \hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta]$$

subject to $\tilde{m}(\theta \mid \eta) \in \Delta(\hat{M})$,

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}(\tilde{m}(\theta' \mid \eta))$$

for any $\theta, \theta' \in \Theta(\eta)$, and

$$\tilde{u}_A(\theta, \eta) \geq u_A(\theta, \eta)$$

for all $\theta \in \Theta(\eta)$. The first constraint states truthful revelation of the agent's true cost to S is consistent with the agent's incentives, and the second constraint requires A to attain a payoff at least as large as what he would expect to attain by playing the grand contract noncooperatively.

Let the maximum payoff of S in the side contracting problem in state η be denoted by $W(\eta)$.

Definition 2 *The (conditional expected) allocation* $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta)) : K \rightarrow \Re^2 \times \Re_+$ *is weakly collusion proof (WCP) if for every* $\eta \in \Pi$*:* $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta, \eta)) = ((\theta, \eta), u_A(\theta, \eta))$ *solves problem* $P(\eta)$ *in which S achieves a maximum payoff of* $W(\eta) = E[u_S(\theta, \eta) \mid \eta]$.

3.4 Characterization of WCP Allocations

We now characterize WCP allocations. This requires us to define a family of 'modified' virtual cost functions, representing the effective cost incurred by the coalition in delivering a unit of output to P, following selection of an optimal side-contract.

Definition 3 *For any* $\eta \in \Pi$*,* $Y(\eta)$ *is a collection of* **coalitional shadow cost (CSC)** *functions* $\pi(\cdot \mid \eta) : \Theta(\eta) \rightarrow \Re$ *which satisfy the following property. For any function in this collection, there exists a real-valued function* $\Lambda(\theta|\eta)$ *which is non-decreasing in* $\theta \in \Theta(\eta)$ *with* $\Lambda(\underline{\theta}(\eta) \mid \eta) = 0$ *and* $\Lambda(\bar{\theta}(\eta) \mid \eta) = 1$*, such that*

$$\pi(\theta|\eta) \equiv \theta + \frac{F(\theta \mid \eta) - \Lambda(\theta \mid \eta)}{f(\theta \mid \eta)} \tag{3}$$

Equation (3) modifies the usual expression for virtual cost $h(\theta|\eta) \equiv \theta + \frac{F(\theta|\eta)}{f(\theta|\eta)}$ by subtracting from it the non-negative term $\frac{\Lambda(\theta|\eta)}{f(\theta|\eta)}$. Intuitively, with collusion between S and A, it is as if P procures the good from a single entity, consisting of the coalition of

S and A. If A's outside option payoff in the side-contracting problem were 0 instead of $u_A(\theta, \eta)$, S would incur a cost of $h(\theta|\eta)$ in arranging for delivery of one unit of the good. P's problem of procuring the good would then reduce to contracting with a single agent with an unknown cost of $h(\theta|\eta)$. This is worse for P compared with the situation where there is no supervisor at all — in the latter context, P would be contracting with A alone who incurs a cost of θ rather than $h(\theta|\eta)$. This is the well-known problem of double marginalization of rents (DMR), arising due to exercise of monopsony power by S in side-contracting with A. As elaborated later, this is why delegating the right to contract (with A) to S cannot result in any improvement for P compared to the situation where no S is employed.

To limit DMR, P contracts with both S and A, and provides A with an outside option (of $u_A(\theta, \eta)$) that effectively raises his bargaining power *vis-a-vis* S while negotiating the side contract. Meeting a larger outside option for A effectively induces S to deliver a higher output to P: this is what paying a higher rent to A necessitates. The extent of DMR is then curbed: the shadow cost for the coalition in delivering a unit of output to P is lowered. This lowering of the virtual cost is represented by the subtraction of the term $\frac{\Lambda(\theta|\eta)}{f(\theta|\eta)}$ from what it would have been ($h(\theta|\eta)$) under delegated contracting. The derivative of $\Lambda(\theta \mid \eta)$ represents the Kuhn-Tucker multiplier on A's (type θ) participation constraint in S's problem of selecting an optimal side contract. Since the multiplier is non-negative, the $\Lambda(\theta \mid \eta)$ function is non-decreasing.

However, $\pi(\theta|\eta)$ is not the correct expression for the shadow cost of output for the coalition, if it is non-monotone in θ. In that case, it has to be replaced by its 'ironed' version (Myerson (1981)), using the distribution function $F(\theta|\eta)$. Let the corresponding ironed version of $\pi(\theta|\eta)$ be denoted by $z(\theta|\eta)$: we call this a *coalitional virtual cost function*.

Definition 4 *For any $\eta \in \Pi$, the set of* **coalitional virtual cost (CVC)** *functions is the set*

$$Z(\eta) \equiv \{z(\cdot \mid \eta) \quad \textit{is the ironed version of some} \quad \pi(\cdot \mid \eta) \in Y(\eta)\}.$$

of functions obtained by applying the ironing procedure to the set $Y(\eta)$ of CSC functions.[16] *Denote by $\Theta(\pi(\cdot \mid \eta), \eta)$ the corresponding pooling region of θ where $\pi(\cdot|\eta)$ is flattened by the ironing procedure.*

As the next result shows, every WCP allocation satisfies coalitional participation and incentive constraints corresponding to some coalitional virtual cost function z. Combined with an individual incentive compatibility constraint for A, and a constraint that output must be constant over regions where the ironing procedure flattens the underlying CSC function, these coalitional constraints characterize WCP allocations.

Proposition 1 *The allocation (u_A, u_S, q) is WCP if and only if the following conditions hold for every η. There exists a CVC function $z(\cdot|\eta) \in Z(\eta)$ such that*

(i) For every $(\theta, \eta), (\theta', \eta') \in K \equiv \{(\theta, \eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$,

$$X(\theta, \eta) - z(\theta \mid \eta)q(\theta, \eta) \geq X(\theta', \eta') - z(\theta \mid \eta)q(\theta', \eta')$$

$$X(\theta, \eta) - z(\theta \mid \eta)q(\theta, \eta) \geq 0$$

where

$$X(\theta, \eta) \equiv u_A(\theta, \eta) + u_S(\theta, \eta) + \theta q(\theta, \eta)$$

(ii) For any $\theta, \theta' \in \Theta(\eta)$,

$$u_A(\theta, \eta) \geq u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta)$$

(iii) $q(\theta, \eta)$ is constant on any interval of θ which is a subset of the corresponding pooling region of the CVC function z.

Condition (i) represents the coalitional incentive and participation constraints corresponding to contracting with a single agent with a unit cost of z. Condition (ii) is the individual incentive compatibility constraint for A. Condition (iii) states that the output must be constant over every interval in the pooling region.

[16] The ironing procedure ensures these functions are continuous and non-decreasing.

3.5 Justification for WCP Allocations

In this section, we provide a justification for focusing attention on WCP allocations.

The notion of Weak Perfect Bayesian Equilibrium (WPBE) of the game with collusion requires beliefs and continuation strategies to be specified corresponding to all information sets of the game.[17] As there are typically multiple WPBEs of the continuation game following any given GC offer, we need to specify how these might be selected.

If the mechanism design problem is stated as selection of an allocation by the Principal subject to the constraint that it can be implemented as the outcome of some WPBE following a choice of a grand contract, it is presumed that the Principal is free to select continuation beliefs and strategies for noncooperative play of the grand contract following off-equilibrium path rejections of offered side contracts by S to A. It can be shown that in such a setting the problem of collusion can be completely overcome by the Principal, with appropriate selection of off-equilibrium-path continuations. This is formally shown in the working paper version of this paper (Mookherjee et al. (2014)). A heuristic description of how the second-best payoff can be achieved by the Principal as a WPBE is as follows. P selects a grand contract and recommends a noncooperative equilibrium of this contract in which (i) conditional on participation by S, noncooperative play results in the second-best allocation; (ii) S is paid nothing; and (iii) if S does not participate, P offers A a 'gilded' contract providing the latter a high payoff in all states. On the equilibrium path S always offers a null side contract. If A rejects any offer of a non-null side-contract, they mutually believe that subsequently S will not participate in the grand contract, and A will receive the gilded contract. This forms a WPBE as rejection of any non-null side contract is sequentially rational for A given A's belief that S will exit following any rejection. And exit is sequentially rational for S given his belief that A will reject the side contract and they will subsequently play the grand contract noncooperatively

[17] For definition of WPBE, see Mas-Colell, Whinston and Green (1995, p.285).

where S will be paid nothing.

Collusion is overcome by the Principal here by exploiting a lack of coordination among A and S over continuation beliefs and play of the side contracting game. This denies the essence of collusive activity, which involves coordination by the colluding parties 'behind the Principal's back'. It is therefore reasonable to insist that S and A can collectively coordinate on the choice of side-contracting equilibria that are Pareto-undominated (for the coalition). Specifically, this rules out WPBE outcomes for which (following some realization of η) there exists some side-contract offer and a PBE of the subsequent continuation game played by S and A which generates a higher expected payoff for S, without lowering the expected payoff of any type of A. Appendix A provides an alternative noncooperative justification for WCP allocations in terms of a restriction on off-equilibrium-path beliefs which generalizes the assumption of 'passive' beliefs which has been employed by many previous authors.

Definition 5 *Following the selection of a grand contract by P, a WPBE(wc) is a Weak Perfect Bayesian Equilibrium (WPBE) of the subsequent game with the following property. There does not exist some signal realization η, and some deviating side-contract offer SC(η) for which there is a Perfect Bayesian Equilibrium (PBE) of the subsequent continuation game in which (conditional on η) S's payoff is strictly higher and A's payoff not lower for any type.*

Definition 6 *An allocation (u_A, u_S, q) is implementable in the weak collusion game if there exists a grand contract and a WPBE(wc) of the subsequent game which results in this allocation.*

We now show that the WPBE(wc) refinement corresponds to WCP allocations that satisfy interim participation constraints. Note that the WPBE(wc) notion allows for collusion to occur (i.e., a non-null side contract to be offered and accepted by some types of A), and also for side-contract offers to be rejected by some types of A. Hence the WCP notion does not rest on any arbitrary restrictions on side contract outcomes, e.g., which rule out the possibility of equilibrium-path rejections by A of the side contract offered by S. The problem discussed by Celik and Peters (2011) therefore does not apply to this setting. Moreover, the restriction to WCP allocations which correspond to equilibrium outcomes in which collusion does not occur on the equilibrium path, is also without loss of generality.

Proposition 2 *An allocation (u_A, u_S, q) is implementable in the weak collusion game, if and only if it is a WCP allocation satisfying interim participation constraints*

$$E[u_S(\theta, \eta)|\eta] \geq 0 \textit{ for all } \eta \tag{4}$$

$$u_A(\theta, \eta) \geq 0 \textit{ for all } (\theta, \eta) \tag{5}$$

4 Main Results

We are now in a position to present our main results. In this section we will compare the following organizational alternatives:

(a) *No Supervisor (NS):* where P does not employ S and contracts with A alone on the basis of his own prior information F over A's cost θ. This is a special case of the preceding model where $X_S \equiv 0, M_S \equiv \emptyset$ in the grand contract. It is well known that P attains an expected profit of $E[V(q^{NS}(\theta)) - H(\theta)q^{NS}(\theta)]$ where $q^{NS}(\theta)$ is defined by the property $V'(q^{NS}(\theta)) = H(\theta)$. We shall denote this profit by Π_{NS}.

(b) *Delegated Supervision (DS):* Here P contracts with S alone, and delegates to S the authority to contract with A and make production decisions. It is a special case of the preceding model where $X_A \equiv 0, M_A \equiv \emptyset$ in the grand contract. S enters into a side-contract with A, and then responds to P's contract offer with a message regarding the joint realization of θ and η, or some summary of the two variables. Here A has no outside option of rejecting the side contract and participating in the grand contract, which increases the bargaining power of S with A. We shall denote the resulting profit of P by Π_{DS}.

(c) *Centralized Supervision (CS):* This is the unrestricted version of the model considered so far, where P offers a grand contract involving both S and A. A now has an outside option of rejecting the side contract offered by S and participating in the grand contract noncooperatively. We shall denote the resulting profit of P by Π_{CS}.

We will also assess these relative to the benchmark of no collusion, which is associated with the second-best allocation defined previously. The associated profit will be denoted Π_{SB}. Since S has access to information about A's cost that is valuable in contracting with A, it is obvious that $\Pi_{NS} < \Pi_{SB}$, i.e., hiring S is valuable if there is no collusion. We now compare the three alternatives above against one another, and will subsequently assess them relative to the second-best.

Proposition 3 $\Pi_{DS} < \Pi_{NS}$: *delegated supervision is worse for the Principal compared to hiring no supervisor.*

The result of Faure-Grimaud, Laffont and Martimort (2003) therefore does not extend to the setting of our model with ex ante collusion, risk neutrality and continuous types. The intuitive reason is simple. Ex ante collusion implies that in contracting with P, the supervisor is subject to an ex post participation constraint: he can accept or reject the contract offered by P *after* he has learnt the realization of A's cost θ. This results in *double marginalization of rents (DMR)*: A earns rents owing to his private information regarding θ with respect to S, and then S earns rents owing to his private information regarding his costs of procuring from A (which depend on the realizations of θ and η). In DS, the Principal effectively contracts with a single agent whose unit cost equals $\hat{h}(\theta|\eta)$ which is the ironed version of $h(\theta|\eta) \equiv \theta + \frac{F(\theta|\eta)}{f(\theta|\eta)}$, who can decide whether to participate after observing the realization of his unit cost.

Since $h(\theta|\eta) > \theta$ almost everywhere (which implies the same is true for its ironed version $\hat{h}(\theta|\eta)$), delegated supervision amounts to contracting with a single supplier whose cost is uniformly higher, compared to contracting with the agent alone in the absence of the supervisor. While it is relatively easy to show that DS cannot dominate NS, the proof establishes the stronger result that DS is **strictly** dominated by NS.[18]

Proposition 4 $\Pi_{NS} < \Pi_{CS}$*: the Principal is strictly better off hiring S and contracting directly with both S and A, compared to hiring no supervisor.*

This states that P always benefits from hiring S despite the presence of ex ante collusion between S and A. Combining with the previous result, it follows that S is valuable only provided P does not delegate authority to S: it is essential that P contracts simultaneously with A as well, thus providing A an outside option which raises A's bargaining power within the coalition. This limits the DMR problem by countervailing S's tendency to behave monopsonistically with respect to A. By raising A's outside option, the coalitional virtual cost z is reduced, allowing an increase in output delivered, and raising P's expected payoff.

This helps explain how contracting directly with both S and A helps reduce the DMR problem inherent in DS which rendered it inferior to NS. However, it does not help explain why it manages to do so sufficiently that CS ends up being superior to NS. The explanation for this is more subtle, arising from P's ability to profitably utilize S's superior information concerning the agent's cost with a simple mechanism. This arises ultimately from the discrepancy between relative likelihoods of different cost states by P and S, which they use to weight different states in computing their respective payoffs.

[18]The proof of strict domination is also straightforward in the case that $h(\theta|\eta)$ is continuous and nondecreasing in θ over a common support $[\underline{\theta}, \bar{\theta}]$ for every η. In that case an argument based on Proposition 1 in Mookherjee and Tsumagari (2004) can be applied. In the general case there are a number of additional technical complications, but the result still goes through.

It may help to outline the WCP allocation that can be used by P. Starting with the optimal allocation in NS (which corresponds to the special case of CS where $\Lambda(\theta \mid \eta)$ is chosen equal to $F(\theta \mid \eta)$, ensuring that the CSC and CVC functions both reduce to the identity function ($\pi(\theta|\eta) = z(\theta|\eta) = \theta$)), P can construct a small variation in the CVC function z in some state η^*, raising it above θ for some interval Θ_H and lowering it for some other interval Θ_L, both of which have positive probability given η^*. The corresponding quantity procured $q(\theta, \eta^*)$ is set equal to $q^{NS}(z(\theta|\eta^*))$, the quantity procured in NS when the agent reported a cost of $z(\theta|\eta^*)$. This corresponds to raising the quantity procured from the coalition over Θ_L and lowering it over Θ_H. Payments to the coalition are set analogously at $X^{NS}(z(\theta|\eta^*))$, what the agent would have been paid in NS following such a cost report.[19] The agent is offered the associated rent: $u_A(\theta, \eta^*) = \int_\theta^{\bar{\theta}} q^{NS}(z(y|\eta^*))dy$. By construction, this allocation satisfies the agent's incentive and participation constraints, as well as the coalitional incentive constraint.[20]

Proposition 1 ensures such an allocation is WCP, provided S's interim participation constraint is satisfied. The variation over Θ_L lowers rents earned by S, and over Θ_H raises them. Since S does not earn any rents to start with (i.e, in NS), it is necessary to construct the variation such that S's expected rents in state η^* do not go down. The rate at which S's rents vary locally in state θ with the quantity procured equals $\frac{F(\theta|\eta^*)}{f(\theta|\eta^*)}$.[21] Intuitively this is the saving that can be pocketed by S when procuring one less unit of the good from A. Maintaining S's expected rent therefore

[19]Specifically, $X^{NS}(z(\theta|\eta)) = z(\theta|\eta)q^{NS}(z(\theta|\eta)) + \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(y)dy$.

[20]This requires checking that there exists a CSC function $\pi(\theta|\eta)$ corresponding to some function $\Lambda(\cdot \mid \eta)$ on $[\underline{\theta}(\eta), \bar{\theta}(\eta)]$ satisfying the requirements in the definition of a CSC function, such that $z(\theta \mid \eta)$ is the ironed version of $\pi(\theta \mid \eta)$. This is true, since we can select $\Lambda(\theta \mid \eta) = (\theta - z(\theta \mid \eta))f(\theta \mid \eta) + F(\theta \mid \eta)$, which is strictly increasing over Θ_L and Θ_H for a sufficiently small variation of z from the identity function. Then $\Lambda(\cdot \mid \eta)$ is a function which satisfies the required properties and generates $\pi(\theta|\eta) = z(\theta \mid \eta)$, since $z(\theta \mid \eta)$ is a non-decreasing function.

[21]S's interim rent in state η equals the expected value conditional on η of $X^{NS}(z(\theta|\eta)) - u_A(z(\theta|\eta)) - \theta q^{NS}(z(\theta|\eta))$, i.e., equals $E[\{z(\theta|\eta) - h(\theta|\eta)\}q^{NS}(z(\theta|\eta)) - \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(z)dz|\eta]$.

implies a marginal rate of substitution between output variations over Θ_L and Θ_H that equals the ratio of the (average) conditional inverse hazard rates $\frac{F(\theta|\eta^*)}{f(\theta|\eta^*)}$ over these two intervals respectively.

On the other hand, P's benefit from a small expansion in output delivered in state θ equals $V'(q^{NS}(\theta)) - \theta$, where $q^{NS}(\theta)$ denotes the optimal allocation in NS.[22] This allocation satisfies $V'(q^{NS}(\theta)) = H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$, the virtual cost of procurement without conditioning on information regarding η. Hence P's marginal benefit from output expansion in state θ equals the unconditional inverse hazard rate $\frac{F(\theta)}{f(\theta)}$. This implies that P's marginal rate of substitution between output variations over Θ_L and Θ_H equals the ratio of the (average) unconditional inverse hazard rates $\frac{F(\theta)}{f(\theta)}$ over these two intervals. The informativeness of S's signals implies that P's marginal rate of substitution differs from S's in some state η^* over some pair of intervals Θ_L, Θ_H. Hence there exist variations of the type described above which raise P's expected payoff, while preserving the expected payoff of S.

One may wonder whether the gains achieved by the Principal from hiring S are marginal rather than substantial. Section 5.4 shows that the second-best payoff is achievable for some cases in the context of variants of the model where the Principal's benefit function is linear. This is consistent with results of Pavlov (2008) and Che and Kim (2009) in the case of auctions (where the indivisibility of the object being auctioned renders the context analogous to a linear benefit function). In the context of nonlinear benefit functions, the following result shows that the second-best is not achievable provided the benefit function exhibits sufficient curvature (besides some standard restrictions on the information structure).

Proposition 5 $\Pi_{CS} < \Pi_{SB}$: *P cannot attain the second-best payoff in CS if the following conditions hold:*

(i) The support of θ does not vary with the signal: $\Theta(\eta) = \Theta$ for any $\eta \in \Pi$;

[22]This follows from the fact that $\frac{\partial X^{NS}(z)}{\partial z} = zq^{NS'}(z)$, implying that the marginal increase in payment evaluated at $z = \theta$ equals θ times the marginal output change.

(ii) there exists $\eta^* \in \Pi$ *such that* $f(\theta|\eta^*)$ *and* $\frac{f(\theta|\eta^*)}{f(\theta|\eta)}$ *are both strictly decreasing in* θ *for any* $\eta \neq \eta^*$*; and*

(iii) $V'''(q) \leq \frac{(V''(q))^2}{V'(q)}$ *for any* $q \geq 0$.

Condition (i) states that the support of θ does not vary with η, while (ii) is a form of a monotone likelihood property: there is a signal realization η^* which is 'better' news about θ than any other realization, in the sense of shifting weight in favor of low realizations of θ. It additionally requires that the conditional density $f(\theta|\eta^*)$ is strictly decreasing in θ, i.e., higher realizations of θ are less likely than low realizations when $\eta = \eta^*$. (ii) is satisfied for instance when θ has a uniform prior and there are just two possible signal values satisfying the standard monotone likelihood ratio property. Condition (iii) is satisfied if V is exponential ($V = 1 - \exp(-rq), r > 0$). It corresponds to the assumption of 'non-increasing absolute risk aversion' of the Principal's benefit function.

The proof develops necessary conditions for implementation of the second best given the distributional properties (i) and (ii). If the outputs are second-best, they must be a monotone decreasing function of the (ironed) virtual cost $\hat{h}(\theta \mid \eta)$ in the second-best setting. If they also satisfy the coalitional incentive constraints, they must be monotone in CVC $z(\theta \mid \eta)$. These conditions imply the existence of a monotone transformation from $\hat{h}$ to z, and enable S's ex post rent to be expressed as a function of $\hat{h}$ alone. Condition (iii) is used to show that this rent function is strictly convex which in turn is used to show that the expected rents of S must be strictly higher in state η^* than any other state.

5 Extensions

5.1 Implementation via Modified Delegation

We now show that the optimal allocation can be implemented by a modified form of delegation, where P communicates and transacts only with S on the equilibrium path. In this arrangement, S is 'normally' expected to contract on behalf of the coalition $\{S, A\}$ with P, sending a joint participation decision and report of the state (θ, η) to P after having entered into a side contract with A. However A has the option of circumventing this 'normal' procedure and asking P to activate a grand contract in which A and S will send independent reports and participation decisions to P. The presence of this option ensures that A has sufficient bargaining power within the coalition; it does not have to be 'actually' used, i.e., on the equilibrium path. This mechanism can implement any implementable allocation as a WPBE(wc) outcome.

The argument is as follows (we omit a formal proof). Take any WCP allocation $(u_S(\theta, \eta), u_A(\theta, \eta), q(\theta, \eta))$ defined on K which satisfies interim participation constraints, and let aggregate payments to the coalition be $X(\theta, \eta) = u_A(\theta, \eta) + u_S(\theta, \eta) + \theta q(\theta, \eta)$. Let the associated grand contract be denoted as follows. The message spaces are $\tilde{M}_S, \tilde{M}_A$, where $\tilde{M}_S = \Pi \cup \{e_S\}$ and $\tilde{M}_A = K \cup \{e_A\}$. Both S and A report η, and A additionally reports θ. P cross-checks the two η reports, and conditional on these agreeing with one another, transfers are set in the obvious way corresponding to the allocation $(u_S(\theta, \eta), u_A(\theta, \eta), q(\theta, \eta))$, e.g., when neither party exits, both report η and A reports θ, $\tilde{X}_S(\theta, \eta) = u_S(\theta, \eta), \tilde{X}_A(\theta, \eta) = u_A(\theta, \eta) + \theta q(\theta, \eta), \tilde{q}(\theta, \eta) = q(\theta, \eta)$, otherwise these are all zero.

This 'original' grand contract can be augmented as follows. A is offered a message space $M_A = \tilde{M}_A \cup \{\emptyset\}$, while S is offered $M_S = \tilde{M}_S \cup K \cup \{e\}$. The interpretation is that if $m_A = \emptyset$, A decides not to communicate directly with P. And if $m_S \in K \cup \{e\}$, S decides to submit a joint report (θ, η) (or else communicates a joint shutdown decision e) to P on behalf of the coalition. The choice of $m_A = \emptyset, m_S \in K \cup \{e\}$ will correspond to the 'normal' delegation mode.

When the normal delegation mode is in operation, i.e., $m_A = \emptyset, m_S \in K \cup \{e\}$, P will communicate and transact with S alone. Hence transfers and output assignments in the augmented mechanism are defined as follows: (X_S, X_A, q) equals $(\tilde{X}_S, \tilde{X}_A, \tilde{q})$ on $\tilde{M}_S \times \tilde{M}_A$, $(0, X(m_S), q(m_S))$ if $m_A = \emptyset, m_S \in K \cup \{e\}$, and $(-T, -T, 0)$ otherwise where T is a large positive number. The last feature ensures that A and S will always coordinate on either the normal delegation mode, or the grand contract.

It is easy to check that this augmented mechanism has a WPBE(wc) where both S and A opt for the normal delegation mode, S offers A a side contract with $m_S(\theta, \eta) = (\theta, \eta) \in K$ and $u_A^*(\theta, \eta) = u_A(\theta, \eta)$ for all (θ, η), which A accepts. To see this note first that if S and A play this augmented grand contract noncooperatively, A will never select $m_A = \emptyset$, since this results in a negative payoff for A no matter what S does. If $m_A = \emptyset, m_S \in K \cup \{e\}$, A is committed to producing a positive quantity while not getting paid anything, while $m_A = \emptyset, m_S \in \tilde{M}_S$ implies $X_A = -T, q = 0$. And given that A does not select $m_A = \emptyset$, neither will S select m_S in $K \cup \{e\}$, owing to the large penalty T for mis-coordination. Rejection of a side contract will effectively result in noncooperative play of the original grand contract.

Hence A has an outside option of earning $u_A(\theta, \eta)$ by rejecting any side contract offered by S. This (along with the fact that the allocation is WCP) implies that the side contract offered by S in equilibrium is optimal for S. The reason is that the outcome of any feasible side contract in the normal delegation mode was also attainable as the outcome of some feasible side contract in the original mechanism.

Proposition 6 *Any implementable allocation with weak collusion can be implemented as a WPBE(wc) outcome of the modified delegation mechanism described above, where P communicates and transacts with S alone on the equilibrium path.*

The reverse pattern of modified delegation, where P communicates only with A on the equilibrium path, also happens to be an alternative way of implementing an optimal WCP allocation. We do not present a formal statement or proof for this result. It implies that the model does not provide any argument for superiority of either form of modified delegation over the other. In the context of legal procedures, this suggests the equivalence of plea bargaining arrangements (where the judge seeks a report from accused party and reserves the right to go to trial should a 'not-guilty' plea be made) with the reverse system where the judge seeks a report from a public prosecutor initially and then decides whether or not to go to trial based on this report. If we were to extend our model to include fixed costs of communication of the Principal with either the supervisor or the agent (but not both), it would provide a way of discriminating between the two alternatives. If for instance communication with S is costless while with A is costly, modified delegation to S will be optimal and will dominate modified delegation to A.

5.2 Side Contracts Designed by a Third Party, and Alternative Allocations of Bargaining Power

We now explain how the preceding results extend when the side contract is designed not by S, but instead by a third-party that manages the coalition and assigns arbitrary welfare weights to the payoffs of S and A respectively. Such a formulation has been used by a number of authors to model collusion, such as Laffont and Martimort (1997, 2000), Dequiedt (2006) and Celik and Peters (2011). An advantage of this approach is that it allows an evaluation of the effects of varying the allocation of bargaining power between colluding partners.

Our results extend to such a setting, under the following formulation of side contracts designed by a third party. We assume the third-party's objective is to maximize a weighted sum of S and A's interim payoffs. The third party designs the side contract after learning the realization of η.[23] Both S and A have the option to

[23] This assumption can be dropped without affecting the results, since it can be shown the third-party can use cross-reporting of η by S and A to learn its true value.

reject the side contract, in which case they play the grand contract noncooperatively.

The notion of WCP allocations is extended as follows. Letting $\alpha \in [0, 1]$ denote the welfare weight assigned by the third-party to A's payoff, the side contract design problem reduces to selecting randomized message $\tilde{m}(\theta \mid \eta)$ and A's payoff $\tilde{u}_A(\theta, \eta)$ to (using the same notation for the formulation $P(\eta)$ of side contracts in Section 3.3):

$$\max E[(1-\alpha)\{\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta)\} + \alpha\tilde{u}_A(\theta, \eta) \mid \eta]$$

subject to $\tilde{m}(\theta \mid \eta) \in \Delta(\hat{M})$,

$$\tilde{u}_A(\theta, \eta) \geq u_A(\theta, \eta)$$

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}(\tilde{m}(\theta' \mid \eta))$$

$$E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta] \geq E[u_S(\theta, \eta) \mid \eta].$$

Besides modifying the objective function, this formulation adds a participation constraint for S. We refer to this as problem $TP(\eta; \alpha)$. The definition of WCP can be extended to WCP(α) by requiring the null side contract to be optimal in $TP(\eta; \alpha)$ for every η.

In Appendix B, we explain how WCP(α) allocations can continue to be justified by a suitable extension of the WPBE(wc) concept to this setting. In order to address the Celik-Peters (2011) problem, side contracts consist of two stages: an initial collusion-participation stage, followed by a reporting or execution stage in the event of both parties agreeing to participate at the first stage. The collusion-participation stage enlarges a dichotomous (exit-participate) message set for each party to a larger message set which includes auxiliary messages for A. At the end of the first stage, S and A observe their respective first stage messages; conditional on both agreeing to participate, they communicate type reports to P at the second stage. The auxiliary first-stage messages enable A to communicate more information to S than is possible with a dichotomous participation decision, and replicate outcomes achievable when side contract offers are rejected by some types of A. This enables attention to be restricted to side contracts which are always accepted on the equilibrium path.

In this setting, the WPBE(wc) notion is extended in the obvious manner: it should never be possible for the third party to deviate to some alternative side-contract whose subsequent continuation game has a PBE which generates a higher payoff for the third-party, without lowering the payoff of S or any type of A. In Appendix B we show that allocations implementable as WPBE(wc) outcomes coincide with the set of WCP(α) allocations.

We now claim that the set of WCP(α) allocations is independent of α. This implies that all our preceding results extend to side contracts designed by a third party.[24]

Proposition 7 *The set of WCP(α) allocations is independent of* $\alpha \in [0, 1]$.

The reasoning is straightforward, so we omit a formal proof. The WCP criterion amounts to the absence of incentive compatible deviations that are Pareto improving for the coalition: this property does not vary with the precise welfare weights. Consider any $\alpha \in (0, 1)$. A given allocation is WCP(α) if and only if there is no other allocation attainable by some non-null side contract which satisfies the incentive constraint for A, and which Pareto-dominates it (for A and S) with at least one of them strictly better off. The same characterization applies to any interior $\alpha' \in (0, 1)$, implying that the set of WCP(α) allocations is independent of $\alpha \in (0, 1)$. The transferability of utility can then be used to show that the set of WCP allocations for interior welfare weights are also the same at the boundary.[25]

[24] Faure Grimaud et al. (2003) provide an analogous result for the case of interim collusion.

[25] If an allocation is WCP(1) but not WCP(α) for some interior α, there must exist a non-null side contract SC^* which allows S to attain a strictly higher payoff, which leaves A's payoff unchanged. Then there exists another feasible non-null side-contract which gives A a slightly higher payoff in all states, which meets S's participation constraint. Hence it is possible to design a feasible side contract that raises A's expected payoff, so the original allocation could not have been WCP(1).

5.3 Altruistic Supervisors

Now consider a different variant, where S offers a side-contract to A, but S is altruistic towards A rather than just concerned with his own income. Suppose S's payoff is $u_S = X_S + t + \alpha[X_A - t - \theta q]$, where $\alpha \in [0, 1]$ is the weight he places on A's payoff. A on the other hand is concerned with only his own income: $u_A = X_A - t - \theta q$.

Our analysis extends as follows. It is easy to check that the expression for coalitional shadow cost is now modified to

$$\pi_\alpha(\theta|\eta) \equiv \theta + (1-\alpha)\frac{F(\theta \mid \eta) - \Lambda(\theta \mid \eta)}{f(\theta \mid \eta)}$$

instead of $\pi(\theta|\eta)$ in Definition 3. In DS, the corresponding expression for the cost of procuring one unit from S is modified from $h(\theta \mid \eta)$ to $h_\alpha(\theta \mid \eta) = \theta + (1-\alpha)\frac{F(\theta|\eta)}{f(\theta|\eta)}$. As long as $\alpha < 1$, this is strictly higher than θ, so DS will still continue to result in a lower profit than NS. The proof that CS dominates NS also goes through *in toto*.

It is interesting to examine the effect of changes in the degree of altruism on P's payoffs. An increase in α lowers S's shadow cost of output in DS $h_\alpha(\theta \mid \eta)$, which benefits P. This is intuitive: the DMR problem becomes less acute with a more altruistic supervisor. Note that with perfect altruism $\alpha = 1$, and the DMR problem disappears: DS then becomes equivalent to NS.

On the other hand, an increase in altruism cannot benefit P in CS. The set of WCP allocations can be shown to be non-increasing in α. Take any WCP allocation corresponding to α: the following argument shows that it is a WCP allocation corresponding to any $\alpha' < \alpha$. Let $z(\theta \mid \eta)$ be the CVC function that is associated with the allocation at α, i.e., it is the ironed version of $\pi_\alpha(\theta|\eta)$ corresponding to some function $\Lambda_\alpha(\cdot|\eta)$ satisfying the stipulated requirements in the definition of CSC functions on $[\underline{\theta}(\eta), \bar{\theta}(\eta)]$. We can then select

$$\Lambda_{\alpha'}(\theta \mid \eta) = \frac{\alpha - \alpha'}{1-\alpha'}F(\theta \mid \eta) + \frac{1-\alpha}{1-\alpha'}\Lambda_\alpha(\theta \mid \eta)$$

when the altruism parameter is α', which satisfies the stipulated requirements since $\alpha > \alpha'$. This ensures that the same CSC and CVC function is available when the altruism parameter is α', since by construction $\pi_\alpha(\theta|\eta) = \pi_{\alpha'}(\theta|\eta)$. Hence the allocation satisfies the sufficient condition for WCP when the altruism parameter is α'.

Finally, if $\alpha = 1$, the CSC function π_α coincides with the identity function θ, the cost of the agent in NS. We thus obtain

Proposition 8 *In CS, P's optimal payoff is non-increasing in α. In DS, P's optimal payoff is increasing in α. When $\alpha = 1$, P's optimal payoffs in DS, NS and CS coincide.*

5.4 Linear Benefit Function

So far we have assumed that V is strictly concave, satisfying Inada conditions so as to guarantee interior allocations. We now briefly describe how preceding results are modified when V is linear upto some capacity limit, and the supervisor's information is represented by a partition. This simple context also helps provide better understanding of the nature of the mechanism design problem and how it can be solved. We present numerical computation of third-best allocations in the case of uniformly distributed costs and a binary information structure, which helps assess the magnitude of benefits from hiring a supervisor despite the presence of collusion.

Let $V(q) = Vq$ with $V \in (\underline{\theta}, \bar{\theta})$ and $q \in [0, 1]$. For simplicity we focus on the case of a binary signal $\eta \in \{\eta_1, \eta_2\}$ where S's information is represented by a partition: $\eta = \eta_1$ represents information that cost is 'low', in which case the true θ lies in the interval $[\underline{\theta}_1, \bar{\theta}_1] = [0, c]$ for some $c \in (0, 1)$. And $\eta = \eta_2$ reveals that cost is 'high': that it lies in $[c, 1]$. Then the conditional distribution functions are $F(\theta \mid \eta_1) = \frac{F(\theta)}{F(c)}$ on $[0, c]$ and $F(\theta \mid \eta_2) = \frac{F(\theta)-F(c)}{1-F(c)}$ on $[c, 1]$. We continue to assume the density $f(\theta)$ is well-defined, continuous and positive everywhere on $[0, 1]$. Define $h_i(\theta)$ and $l_i(\theta)$ as $h_i(\theta) \equiv \theta + \frac{F(\theta|\eta_i)}{f(\theta|\eta_i)}$ and $l_i(\theta) \equiv \theta + \frac{F(\theta|\eta_i)-1}{f(\theta|\eta_i)}$ for $i \in \{1, 2\}$. These are upper and lower bounds for coalitional virtual costs, corresponding to the lowest and highest possible values of the multiplier associated with the agent's outside option when bargaining over the side contract with the supervisor. These reduce to $h_1(\theta) = \theta + \frac{F(\theta)}{f(\theta)}$, $l_1(\theta) = \theta + \frac{F(\theta)-F(c)}{f(\theta)}$, $h_2(\theta) = \theta + \frac{F(\theta)-F(c)}{f(\theta)}$ and $l_2(\theta) = \theta + \frac{F(\theta)-1}{f(\theta)}$. To avoid technical problems associated with the need to iron the coalitional virtual cost functions, we confine attention to the case where $H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$ is increasing in θ on $[0, 1]$, and $h_i(\theta)$ and $l_i(\theta)$ are strictly increasing in θ on $[\underline{\theta}_i, \bar{\theta}_i]$ for any $i \in \{1, 2\}$ where $\underline{\theta}_1 = 0$, $\bar{\theta}_1 = \underline{\theta}_2 = c$ and $\bar{\theta}_2 = 1$. This assumption is automatically satisfied in the case of the uniform distribution $F(\theta) = \theta$. We also confine attention to mechanisms not involving any randomization.[26]

Using the general characterization of feasible mechanisms established earlier in the paper, it is easy to show that the Principal's choice reduces to selecting: (i) a total payment X_0 to the coalition in the event that the good is not delivered; (ii) an additional bonus b when it is delivered; and (iii) cost thresholds $\theta_i, i = 1, 2$ where $\theta_1 \in [0, c]$ and $\theta_2 \in [c, 1]$ where the agent delivers the good in state η_i if and only if $\theta < \theta_i$. Let p_1 denote $F(c)$, and p_2 denote $1 - F(c)$. P's maximization problem reduces to

$$\max[V - b][p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)] - X_0$$

subject to

$$X_0 \geq F(\theta_i \mid \eta_i)[\theta_i - b] \quad \text{for } i \in \{1, 2\} \tag{6}$$

$$X_0 \geq 0 \tag{7}$$

and (θ_1, θ_2, b) satisfies

$$\text{If} \quad \theta_i \in (\underline{\theta}_i, \bar{\theta}_i), l_i(\theta_i) \leq b \leq h_i(\theta_i) \tag{8}$$

$$\text{If} \quad \theta_i = \underline{\theta}_i, b \leq \underline{\theta}_i \tag{9}$$

[26] In the case where V is strictly concave, this assumption entails no loss of generality. We are not sure whether the same is true in this context as well.

$$\text{If} \quad \theta_i = \bar{\theta}_i, b \geq \bar{\theta}_i. \tag{10}$$

The cost threshold θ_i ends up being the 'price' that S offers to A for supplying the good, following signal η_i. Hence (6) represents S's participation constraint in this state, requiring that the fixed payment X_0 must be sufficient to cover the expected 'net' cost of paying A (after taking into account the bonus received from P for delivering the good). Condition (7) represents the constraint that collusion is *ex ante*. If it were not satisfied, the coalition would choose to exit in the event that A reported a cost above the offered price θ_i. In the case of *interim* collusion, this condition would not be imposed: S would have to commit to participating before hearing a cost report from A, whence (6) would suffice to ensure S's participation. Hence *ex ante* collusion represents a kind of 'limited liability' constraint.

The remaining three conditions (8, 9, 10) represent coalitional incentive constraints: it must be in S's interest to offer the price θ_i upon observing η_i.[27] All that is needed (for an 'interior' price) is that the bonus b lie somewhere in-between the upper and lower bounds on coalitional virtual cost (modified in an obvious way for non-interior prices). As shown previously, any price offer lying within these bounds can be induced by P by offering suitable outside options to A.

The case of unconditional delegation corresponds to constraining b to equal the upper bound $h_i(\theta_i)$. This is dominated by P contracting with A in the absence of the supervisor, whence b is constrained to equal θ_i. When P contracts with both S and A, b can be lowered further, up to $l_i(\theta_i)$. Figure 1 provides an illustration of a feasible mechanism where $h_i(\theta_i) > \theta_i > b$ for $i = 1, 2$, in which a given bonus b allows a higher probability of supply in each state than would result with unconditional delegation,

[27] We use here the fact that coalition incentive compatibility requires that the good will be delivered in state η_i if and only if the bonus b exceeds $z(\theta|\eta_i)$, where $z(\theta|\eta_i) = \theta + \frac{F(\theta|\eta_i) - \Lambda(\theta|\eta_i)}{f(\theta|\eta_i)}$ is the coalitional virtual cost function, where $\Lambda(\theta \mid \eta_i)$ is a non-decreasing function taking value 0 and 1 at the endpoints $\underline{\theta}_i$ and $\bar{\theta}_i$ respectively. Hence conditions (8, 9, 10) are necessary. Conversely, given these three conditions, we can find a coalitional virtual cost function satisfying coalition incentive compatibility.

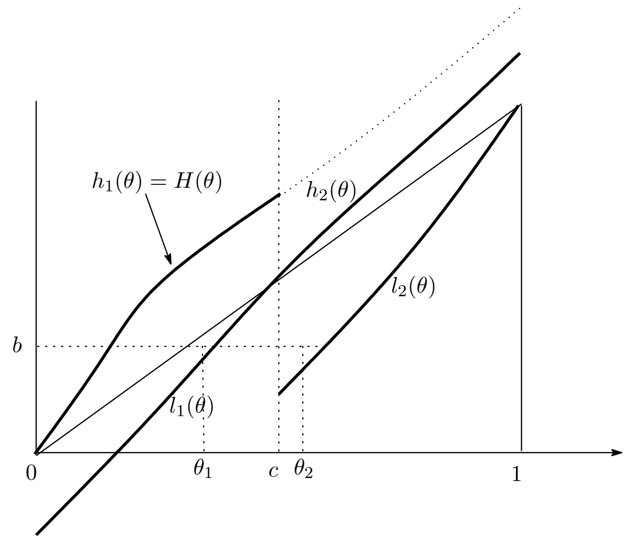

Figure 1: An illustration of a feasible mechanism.

or if P were to not hire S. Of course, raising θ_i above b comes at a cost: a positive fixed payment X_0 has to be made to ensure S's participation constraint (6).

We first examine in this context whether hiring a supervisor is strictly valuable. It makes sense to exclude cases where $V \leq c$, where hiring S is not valuable even in the absence of collusion.[28] Hence we focus on the case where $V > c$, where hiring S is strictly valuable in the absence of collusion.[29]

Proposition 9 *Suppose $V > c$, so hiring S is strictly valuable in the second-best situation. In the presence of weak ex ante collusion, there exists an interval (V_1, V_2) with $V_1 \geq c$, such that hiring S is strictly valuable if and only if $V \in (V_1, V_2)$. $V_1 > c$ if and only if $H(\max\{0, l_2(c)\}) > c$, while $V_2 \geq H(1)$.*

[28]If $V \leq c$, the second-best with a honest supervisor involves zero probability of procurement in state η_2, and offering A a price of θ_1^{SB} which satisfies $V = H(\theta_1^{SB}) = h_1(\theta_1^{SB})$ in state η_1. This can be implemented by P offering A a price of θ_1^{SB} irrespective of η_i; hence S is not needed.

[29]With $V > c$, P will procure with positive probability in state η_2 and the second-best price offered to A will necessarily differ between the two states η_1, η_2, since the price offered in state η_1 will not exceed c while it will exceed it in state η_2. Hence S is valuable in the second-best situation.

This result shows that in contrast to previous Sections with divisible quantities and a strictly concave benefit function, collusion may destroy the value of supervision in some circumstances.[30] This can happen for instance when P's benefit from the good V is very large, so she ends up procuring with probability one in either state η_i in the third-best outcome. This is only possible if P offers to pay the maximum cost of 1 for delivery in either state η_i.[31] Owing to collusion, it is no longer possible to offer a lower price in state η_1 and still guarantee delivery.

For lower values of V where the good may not be delivered with positive probability, the result is less obvious. Proposition 9 states that the condition $H(\max\{0, l_2(c)\}) \leq c$ is sufficient to ensure hiring S is strictly valuable for all values of V slightly above c. This can be explained as follows. In the absence of S, P would offer a price θ^{NS} below c, if V lies between c and $H(c)$. Then P would procure the good with zero probability in state η_2. This corresponds to the allocation $\theta_1 = \theta^{NS} = b, \theta_2 = c, X_0 = 0$. Upon hiring S, P can offer the following allocation which would generate a strict improvement. θ_1 could be left unchanged at θ^{NS}, while $\theta_2^{'}$ could be raised slightly above c. See Figure 2. This enables the delivery probability to be increased in state η_2 and left unchanged in state η_1. For $\theta_2^{'}$ close enough to c, it is true that $F(\theta_2^{'} \mid \eta_2) < F(\theta^{NS} \mid \eta_1)$. Hence a contract $(X_0^{'}, b^{'})$ can be chosen to satisfy

$$X_0^{'} = F(\theta^{NS} \mid \eta_1)(\theta^{NS} - b^{'}) = F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - b^{'}).$$

where the bonus $b^{'}$ is now slightly lower than before, satisfying the following condition, $\max\{l_1(\theta^{NS}), l_2(\theta_2^{'})\} \leq b^{'} < \theta^{NS}$. Given that $H(max\{0, l_2(c)\}) \leq c < V(= H(\theta^{NS}))$ implies $l_2(c) < \theta^{NS}$ then for $\theta_2^{'}$ close enough to c, $l_2(\theta_2^{'}) < \theta^{NS}$, making this choice of $b^{'}$ possible. Then P benefits as S continues to earn zero rent in either state, while moving the allocation closer to the second-best.

The condition $H(\max\{0, l_2(c)\}) \leq c$ turns out to also be necessary to ensure

[30] It can be shown that if the participation constraint (6) for S is strengthened to hold *ex post* rather than *interim*, then supervision ceases to be valuable. This is in contrast to the case where the benefit function is strictly concave, whence it may be possible in some circumstances to hire a

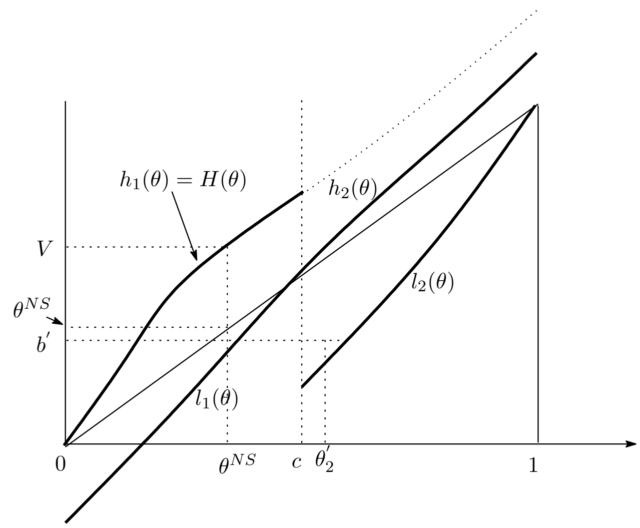

Figure 2: Value of Supervisor.

a strict value of supervision for values of V slightly above c. The proof of this is somewhat involved (see the Appendix), but the underlying idea is the following. Suppose $H(\max\{0, l_2(c)\}) > c$, implying $l_2(c) > \theta^{NS}$ for V close enough to c. An improvement over no-supervision would require P to procure with positive probability in state θ_2. This requires raising the bonus b' above $l_2(c)$, which is higher than $b = \theta^{NS}$. Correspondingly, the optimal θ_1 also needs to be raised discontinuously, which lowers profits of P in state η_1. If V is sufficiently close to c, the increased profits in state η_2 are negligible, and cannot outweight the losses in state η_1.

Part of the reason that the value of supervision is lower in the linear benefit case is that the set of controls available to P are limited: e.g., there is no scope for varying the level of provision. On the other hand, with linear benefits we can show that there exist a range of parameter values where the benefits of hiring S are substantial: the second-best payoff can be achieved.

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supervisor even with *ex post* participation constraints.

[31]If $\theta_i = \bar{\theta}_i$, condition (10) requires $b \geq \bar{\theta}_i$. Hence $b = 1$.

Proposition 10 *Suppose that $V > c$. The second-best payoff can be achieved by P in the presence of collusion if and only if $F(\theta_1^{SB} \mid \eta_1) > F(\theta_2^{SB} \mid \eta_2)$ and*

$$\max\{l_1(\theta_1^{SB}), l_2(\theta_2^{SB})\} \leq \frac{\theta_1^{SB} F(\theta_1^{SB} \mid \eta_1) - \theta_2^{SB} F(\theta_2^{SB} \mid \eta_2)}{F(\theta_1^{SB} \mid \eta_1) - F(\theta_2^{SB} \mid \eta_2)}. \tag{11}$$

where θ_i^{SB} denotes the second-best solution. In the case of a uniform distribution $F(\theta) = \theta$ and $c = 1/2$, this condition reduces to $1/2 < V \leq 3/4$.

The underlying argument is straightforward. Implementation of the second-best allocation entails setting $\theta_i = \theta_i^{SB}$, and ensuring that S earns zero rent in each state. This requires existence of X_0, b such that

$$X_0 = F(\theta_1^{SB} \mid \eta_1)[\theta_1^{SB} - b] = F(\theta_2^{SB} \mid \eta_2)[\theta_2^{SB} - b] \geq 0 \tag{12}$$

for which it is necessary that $F(\theta_1^{SB} \mid \eta_1) > F(\theta_2^{SB} \mid \eta_2)$, and b is set equal to the right-hand-side of (11). Since $\theta_i^{SB} \geq b$, this allocation is feasible if condition (11) is satisfied.

This argument indicates, however, that implementation of the second-best will be generically impossible if there are three or more possible signals observed by S. For example, with three signals, in order to ensure S earns zero rent for all η_i, there must exist b such that

$$F(\theta_1^{SB} \mid \eta_1)[\theta_1^{SB} - b] = F(\theta_2^{SB} \mid \eta_2)[\theta_2^{SB} - b] = F(\theta_3^{SB} \mid \eta_3)[\theta_3^{SB} - b] \geq 0.$$

which requires

$$B(\theta_1^{SB}, \theta_2^{SB}) = B(\theta_2^{SB}, \theta_3^{SB})$$

where

$$B(\theta_i, \theta_j) \equiv \frac{\theta_i F(\theta_i \mid \eta_i) - \theta_j F(\theta_j \mid \eta_j)}{F(\theta_i \mid \eta_i) - F(\theta_j \mid \eta_j)}.$$

This condition will not hold generically.

In the case of strictly concave $V(q)$, our result concerning the impossibility of the second best allocation under suitable conditions was based on a different kind of

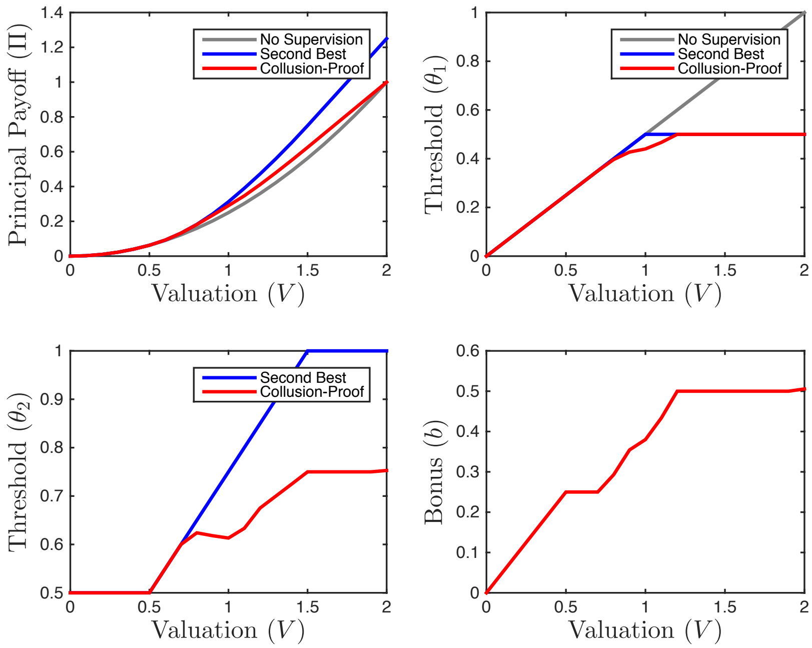

Figure 3: Comparing Optimal Collusion-Proof, Second-Best and No Supervision Allocations, with uniformly distributed cost and $c = \frac{1}{2}$

argument, relying on the continuity of the second best output schedule. With linear benefits, such arguments do not apply as the second best output schedule q^{SB} jumps discontinuously from 1 to 0 at certain points.

It is interesting to note an implication of Proposition 10: second-best implementation requires the good not be procured with positive probability in states η_1 and η_2, which in turn requires V to not be too large. This is similar to the result of Pavlov (2008) and Che and Kim (2009) in the context of auctions, whence second-best implementation requires trade to not occur with positive probability.

With a uniform distribution and $c = 1/2$, we can numerically compute optimal allocations under the second-best, third-best and no-supervision respectively. The results are shown in Figure 3. As shown above, the second best allocation can be implemented in the case $1/2 < V \leq 3/4$. Hiring S is valuable if V is between $3/4$ and 2. Compared to the second-best, we see that for some intervals of V between

3/4 and 2 the probability of procurement decreases, especially in state η_2.

6 Concluding Comments

We have analyzed implications of weak ex ante collusion between a supervisor and agent, where collusion arises with regard to both participation and reporting decisions, and outside option payoffs in coalitional bargaining are determined by noncooperative equilibria of a grand contract designed by the Principal. We showed in such settings that the Principal can still benefit from employing the supervisor. This requires the Principal to design a grand contract involving both the supervisor and the agent, rather than delegating authority over contracting with the agent to the supervisor in an unconditional manner. It is essential for the Principal to give both parties suitable outside option payoffs by designing such a grand contract judiciously. The presence of such a centralized safeguard as an option then allows optimal outcomes to be implemented by delegating authority to the supervisor. These results are consistent with the widespread prevalence of delegation to information intermediaries, and highlight the importance of centralized oversight mechanisms that are needed to mitigate their 'abuse of power'. While the commonsense justification for such mechanism is typically based on considerations of fair treatment of agents, our analysis shows how such mechanisms are essential to prevent inefficient output contractions and loss of profits of the Principal owing to monopsonistic behavior by intermediaries to whom authority is delegated.

In future research, we plan to explore implications of various notions of strong collusion, where one or more of the colluding partners are 'powerful' in the sense of being able to commit how they would behave in the event that someone vetoes a coalitional proposal.

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Appendix A: Justification for WCP Allocations in Terms of Belief Restrictions

In the text we provided a justification of WCP allocations in terms of the equilibrium refinement WPBE(wc), which incorporated a notion of collusion wherein S and A can collectively coordinate on choice of a PBE following any given side-contract. We now provide an alternative 'noncooperative' justification, in terms of restrictions on off-equilibrium-path beliefs alone. This generalizes the assumption of *passive beliefs* often made in the literature (e.g., Faure-Grimaud, Laffont and Martimort (2003)). The 'passive beliefs' assumption requires beliefs following rejection of side-contract offers to not vary with the side-contract offered. Such a restriction rules out implementation of the second-best along the argument in Section 3.5.

Faure-Grimaud, Laffont and Martimort (2003), however, restrict attention to side contracts offered that are always accepted by A on the equilibrium path. Celik and Peters (2011) have shown in the context of a model of a two-firm cartel that this restriction may entail a loss of generality. In contrast to a standard principal-agent setting where agent outside options are exogenous, the consequences of rejection of a side-contract subsequently results in A and S playing a noncooperative game and are thus endogenous. Rejection of a side contract by some types of A can communicate information to S about A's type, affecting subsequent play and resulting payoffs in the noncooperative game. Celik and Peters demonstrate collusive allocations amongst cartel members which can only be supported by side-contract offers which are rejected with positive probability on the equilibrium path.

To address this problem, we allow for side contract offers that might be rejected by some types of A and accepted by others. This is combined with the following restriction on beliefs.

Definition 7 *A WPBE(w) is a Weak Perfect Bayesian Equilibrium (WPBE) satisfying the following restriction on beliefs (conditional on realization of any η): (a)* *there is a pair of beliefs $p(\eta)$ and Bayesian equilibrium $c(\eta) \in C(p(\eta); \eta)$ which results in the noncooperative play of the grand contract following rejection of* **any** *non-null side contract offered by S, where (b) $(p(\eta), c(\eta)) = (p_\emptyset(\eta), c_\emptyset(\eta))$ if S offers a null side-contract on the equilibrium path.*

Criterion (a) imposes the restriction that there is a common continuation belief and Bayesian equilibrium of the grand contract, following rejection of *any* non-null side-contract.[32] Criterion (b) additionally requires this continuation to be the same as the continuation that results when S offers a null side-contract on the equilibrium path.[33] In this case, the consequences of rejection are independent of the side contract offered, and are taken as given by the Principal.

One could argue that it would be reasonable to expand the scope of (b) and also require $(p(\eta), c(\eta)) = (p_\emptyset(\eta), c_\emptyset(\eta))$ whenever a non-null SC is offered and accepted by all types of A on the equilibrium path. Evidently, the definition of WPBE(w) is consistent with this stronger version of (b). However, it is not needed for the results that follow. The Faure-Grimaud, Laffont and Martimort (2003) assumption of passive beliefs (where rejection of any offered SC is followed by beliefs $(p_\emptyset(\eta), c_\emptyset(\eta))$) is therefore consistent with WPBE(w). Their approach can be rationalized by an underlying restriction to side contract offers that are either accepted by all types, or rejected by all types. So WPBE(w) may be viewed as a generalization of the assumption of passive beliefs, when one allows rejection of SCs by some types on the equilibrium path.

We now show that with this restriction on beliefs, there is no loss of generality in confining attention to side-contract offers that are accepted by all types on the

[32] This is irrespective of whether or not rejection occurs on the equilibrium path. If it does, whereby subsequent continuation beliefs are determined by Bayes Rule, (a) requires the same beliefs to ensue from rejection of some other non-null SC.

[33] Criterion (a) by itself is insufficient to allow collusion to have any bite, since the construction used in the argument of Section 3.5 satisfied (a). Hence part (b) is additionally required to avoid its conclusion.

equilibrium path.

Lemma 1 *Given any grand contract, and any allocation resulting from a WPBE(w) in which S's side contract offer is rejected with positive probability on the equilibrium path, there exists another WPBE(w) resulting in the same allocation in which the side contract offered by S is accepted by all types of A on the equilibrium path.*

Proof of Lemma 1: Suppose on the equilibrium path S offers a side contract SC^* in some state $\eta \in \Pi$ which is rejected by a set $T_r \subseteq \Theta(\eta)$ of types of A with positive measure conditional on η. Let the continuation beliefs following rejection of SC^* be denoted p^*, and the Bayesian equilibrium of the grand contract thereafter is denoted $c^* \in C(p^*)$ (here we are suppressing η in the notation for expositional convenience).

Now suppose S offers an alternative side contract $\tilde{SC}$, which agrees with SC^* if A reports $\theta \in \Theta(\eta)\backslash T_r$ to S, i.e., results in the same coordinated report to P and the same side-payment as stipulated by SC^*. If instead A reports $\theta \in T_r$, S proposes the same joint report (θ, η) they would have made independently in c^*, with no side-payment. If $\tilde{SC}$ is rejected by A, they play according to (p^*, c^*) in the grand contract. This ensures consistency with criteria (a) and (b) in the definition of WPBE(w).

If all types of A accept $\tilde{SC}$ and report truthfully, it results in the same allocation as in SC^*. Rejecting it results in the same continuation play of the grand contract that resulted from rejecting SC^*. Conditional on accepting $\tilde{SC}$, no type θ of A can benefit from deviating from truthful-reporting. Otherwise, if $\theta \in \Theta(\eta)\backslash T_r$ benefitted from deviating, this would imply they would have had a profitable deviation from their equilibrium response to SC^*. If $\theta \in T_r$ benefits by deviating, this type would have benefitted earlier also, either by accepting SC^*, or rejecting it and then deviating to the strategy played by some other type of A while playing the Bayesian equilibrium of the grand contract.

Owing to restriction (a) of Definition 7, rejection of any other side-contract offer SC' will also result in the same continuation outcomes in the grand contract. Hence the consequences of S deviating to some other side contract offer remain unchanged. The consequences of not offering a side contract have not changed. So it is optimal for S to offer $\tilde{SC}$. ■

The argument resembles the standard one underlying the Revelation Principle: offering a new side-contract $\tilde{SC}$ which mimics the outcomes resulting from rejection of an original side-contract (SC), can result in acceptance by all types of A and the same resulting allocation. How can this be reconciled with the Celik-Peters (2011) demonstration of a collusive allocation for a two-firm cartel which is the outcome of a side-contract that is rejected with positive probability in equilibrium, which cannot be achieved by some other side contract that is not rejected on the equilibrium path? There are two main differences between our respective formulations of side-contracting. First, in our model S rather than some third-party offers the side-contract. In the latter case, a participation constraint for S has to be respected. In our model S offers the SC, so there is no need to incorporate a participation constraint for S. However this difference would disappear in the version of our model to be considered in Section 5.2, where side contracts are designed and offered by a third party. The second reason is the WPBE(w) restriction we have imposed. The construction of the example in Celik-Peters (2011, Section 2) hinges on beliefs following rejection that vary with the side-contract in question, contrary to what WPBE(w) requires.[34]

[34]To elaborate further, their example rests on the following feature. Rejection of the side contract analogous to our $\tilde{SC}$ (by the uninformed party) results in coalition members playing the grand contract noncooperatively with beliefs $p_\emptyset$, which differs from beliefs following rejection of the equilibrium side contract. If the two side contracts were associated with the same post-rejection continuation beliefs, the argument underlying Lemma 1 would apply, implying that the $\tilde{SC}$ contract would support the same allocation as the equilibrium side-contract. Their construction is based on the implicit assumption that the designer of the side-contract will disclose information regarding the type reported by the other party for some side contracts (e.g., the equilibrium side contract), and not others (e.g., $\tilde{SC}$) when a given party is the only one to reject the side contract.

The next step is to observe that the *collusion-proofness principle* — that P can without loss of generality restrict attention to noncooperative equilibria of grand contracts that do not provide S with an incentive to offer a non-null side contract — also holds for WPBE(w) allocations.

Lemma 2 *An allocation* (u_A, u_S, q) *is a WPBE(w) outcome if and only if there exists a grand contract GC satisfying the following two properties:*

(i) In any state $\eta \in \Pi$*: participation and truthful reporting by all types of S and A constitutes a Bayesian equilibrium relative to beliefs* $p_{\emptyset}(\eta)$ *obtained by updating on* η *alone, which results in state-*η *allocation:* $(u_A(\cdot, \eta), u_S(\cdot, \eta), q(\cdot, \eta))$*;*

(ii) there is a WPBE(w) of the resulting side-contracting game in which S offers no side-contract for any $\eta \in \Pi$*.*

The argument is straightforward. Lemma 1 ensures that without loss of generality attention can be focused on WPBE(w) in which the equilibrium side contract, if offered in any state η, is not rejected by any type of A. Then there is no room for further coordination by S and A which improves the expected payoff of S while meeting A's acceptance and incentive constraint. If the resulting allocation were offered directly in the grand contract, there would be no scope for S to benefit from any further side-contract.

Lemma 2 implies that allocations achieved as WPBE(w) outcomes following any grand contract coincide with WCP allocations satisfying interim participation constraints for both A and S.

Proposition 11 *An allocation* (u_A, u_S, q) *is a WPBE(w) outcome following some grand contract, if and only if it is a WCP allocation satisfying interim participation constraints*

$$E[u_S(\theta, \eta)|\eta] \geq 0 \quad \textit{for all} \quad \eta \tag{13}$$

$$u_A(\theta, \eta) \geq 0 \quad \textit{for all} \quad (\theta, \eta) \tag{14}$$

Proof of Proposition 11:

Necessity follows straightforwardly from Lemmas 1 and 2. To show sufficiency, consider a WCP allocation satisfying participation constraints. Let P offer the following revelation mechanism in the grand contract: $X_S = X_A = q = 0$ if $m_A = e_A$ or $m_S = e_S$. If $m_A \neq e_A$ and $m_S \neq e_S$, and A reports (θ, η_A) while S reports η_S, $q((\theta, \eta_A), \eta_S) = q(\theta, \eta_S), X_S((\theta, \eta_A), \eta_S) = u_S(\theta, \eta_A), X_A((\theta, \eta_A), \eta_S) = \theta q(\theta, \eta_S) + u_A(\theta, \eta_S) - T(\eta_S, \eta_A)$ where T equals zero if $\eta_A = \eta_S$ and $(\theta, \eta_A) \in K$, and a large negative number otherwise. We first show property (i) of Lemma 2 holds. Consider any η. Conditional on both S and A participating, it is optimal for S to report $\eta_S = \eta$ since S's payoff does not depend on η_S. Given that S is reporting truthfully, it is optimal for A to report $\eta_A = \eta$. WCP implies that the null side contract is feasible in the side contracting problem for every η, hence it is optimal for A to report θ truthfully, given that η is being reported truthfully. Given that both S and A report truthfully conditional on participation, the interim participation constraints imply it is optimal for them to always participate.

Let this equilibrium be denoted c^*. We claim that there is a WPBE(w) in which S always offers a null side contract, whose outcome is c^*. The WPBE(w) restriction implies c^* must be the consequence of rejection by A of any offered non-null side contract. Hence $u_A(\theta, \eta)$ is the outside option of A which S takes as given while selecting a side contract. Since the allocation resulting from c^* is WCP, S cannot benefit from offering any non-null side contract. ■

Appendix B: Justification for WCP Allocations when Side-Contracts are Designed by a Third Party

Here we explain how WCP allocations can continue to be justified when side contracts are offered by a third party, extending the 'cooperative' refinement used in the text for the case where they are offered by S. To address the problem highlighted by Celik and Peters (2011), the side-contract is now modelled as a two stage game played by S and A. The first stage is a 'participation' stage where they communicate their participation decisions in the side contract, in addition to some auxiliary messages in the event of agreeing to participate. The role of these messages is to allow A to signal information about his type while agreeing to participate, which can help replicate whatever information is communicated by side-contract rejection in a setting where communication concerning participation decisions is dichotomous. A and S observe the messages sent by each other at the end of the first stage. At the second stage, A and S submit type reports, conditional on having agreed to participate at the first stage.

Let (D_A^p, D_S^p) denote the message sets of A and S at the participation stage (or p-stage). $e_A \in D_A^p$ and $e_S \in D_S^p$ are the exit options of A and S respectively.

What occurs at the second stage ('execution' or e-stage) depends on $d^p = (d_A^p, d_S^p)$ chosen at the first stage.

- If $d_A^p \neq e_A$ and $d_S^p \neq e_S$, A and S select $(d_A^e, d_S^e) \in D_A^e(d^p) \times D_S^e(d^p)$ respectively. The report to P is selected according to $\tilde{m}(d^p, d^e) \in \Delta(M_A \times M_S)$, associated with the transfers to A and S, $t_A(d^p, d^e)$ and $t_S(d^p, d^e)$ respectively. Owing to wealth constraint of the third party, these are constrained to satisfy $t_A(d^p, d^e) + t_S(d^p, d^e) \leq 0$.

- If either $d_A^p = e_A$ or $d_S^p = e_S$, A and S play GC non-cooperatively.

Given GC and η, the third party decides whether to offer a side-contract $SC(\eta)$

or not (i.e., offer a null side-contract NSC). If a non-null side-contract is offered, A and S play a game denoted by $GC \circ SC(\eta)$ with two stages as described above. On the other hand, if the third party offers a null side-contract NSC at the first stage, A and S play GC non-cooperatively based on prior beliefs $p_0(\eta)$. The third-party's objective is to maximize $E[\alpha u_A(\theta, \eta) + (1-\alpha)u_S(\theta, \eta) \mid \eta]$ in state η.

The refinement WPBE(wc) introduced in the text for the case where the side contract is offered by S, can now be extended as follows.

Definition 8 *Following the selection of a grand contract by P, a WPBE(wc) is a Weak Perfect Bayesian Equilibrium (WPBE) of the subsequent game in which side-contracts are designed by a third party, which has the following property. There does not exist some η, and some deviating side-contract offer SC(η) for which there is a Perfect Bayesian Equilibrium (PBE) of the subsequent continuation game in which (conditional on η) the third-party's payoff is strictly higher, while the payoffs of S and every type of A is not lower.*

Definition 9 *An allocation (u_A, u_S, q) is implementable in the weak collusion game with side contracts designed by a third party assigning welfare weight α to A, if there exists a grand contract and a WPBE(wc) of the subsequent side contract game which results in this allocation.*

Lemma 3 *An allocation (u_A, u_S, q) is implementable in the weak collusion game with side contracts designed by a third party assigning welfare weight α to A, if and only if it is a WCP(α) allocation satisfying the interim participation constraints $u_A(\theta, \eta) \geq 0$ and $E[u_S(\theta, \eta) \mid \eta] \geq 0$.*

Proof of Lemma 3

Proof of Necessity

For some GC, suppose that allocation (u_A, u_S, q) is implemented in the game with weak collusion. Suppose the allocation is achieved as the outcome of a WPBE(wc) in which a non-null side contract $SC^*(\eta)$ is offered on the equilibrium path in some state η, which is rejected by some types of A. We show it can also be achieved as the outcome of a WPBE(wc) in which a non-null side contract is offered in state η and accepted by all types of A. Let Θ_r be the set of types who reject $SC^*(\eta)$. Following A's rejection ($d_A^p = e_A$), suppose that A and S play the grand contract GC based on S's updated belief $p(\cdot \mid \Theta_r, \eta)$. Since we are using the PBE as the solution concept, these beliefs do not depend on S's participation decision. Similarly in the event that A accepts, but S rejects $SC^*(\eta)$, A and S play the grand contract GC based on S's updated belief conditioned on the observation of $d_A^p \neq e_A$. Let $d_A^{p*}(\theta, \eta)$ denote A's decision (on the equilibrium path) at the participation stage. Denoting these beliefs by $p(d_A^p) \equiv p(\cdot \mid d_A^{p*}(\theta, \eta) = d_A^p, \eta)$, S's expected payoff from rejecting $SC^*(\eta)$ is

$$E[u_S(\theta, \eta, c(p(d_A^{p*}(\theta, \eta)))) \mid \eta]$$

where $u_S(\theta, \eta, c) \equiv X_S(m_A(\theta, \eta), m_S(\eta))$ for $c = (m_A(\theta, \eta), m_S(\eta))$.

Now construct a new side-contract $\tilde{SC}(\eta)$ which differs from $SC^*(\eta)$ only in that A's message set at the participation stage is $D_A^p \cup \{\tilde{d}_A^p\}$ instead of D_A^p, and A's choice of $\tilde{d}_A^p$ results in A and S playing of $c(p(e_A))$ in GC without any transfers. It is easily verified that the continuation game $GC \circ \tilde{SC}(\eta)$ has a PBE where no type of A rejects the side-contract, realizing the same allocation (u_A, u_S, q) in an equilibrium. In this equilibrium, type $\theta \in \Theta_r$ reports $d_A^p = \tilde{d}_A^p$ instead of $d_A^p = e_A$. In the off-equilibrium-path event that A rejects $\tilde{SC}(\eta)$, A and S play the grand contract based on the belief $p(\cdot \mid \Theta_r, \eta)$. Since S receives the same information from A's decision about d_A^p, he does not have an incentive to change his decision at the second stage; this in turn implies he has no incentive to change his decision at the participation stage. Since the original equilibrium was a WPBE(wc), so is the newly constructed equilibrium.

Next we show that if allocation (u_A, u_S, q) is realized in a WPBE (wc) in which side contracts are not rejected on the equilibrium path, it must be a WCP(α) allocation. Suppose not: the allocation resulting from some non-null side contract

$(\tilde{u}_A^*(\theta,\eta),\tilde{m}^*(\theta,\eta)) \neq (u_A(\theta,\eta),(\theta,\eta))$ solves the problem $TP(\eta;\alpha)$ for some η. Define $\tilde{u}_S^*(\theta,\eta) \equiv \hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - \tilde{u}_A^*(\theta,\eta)$. It is evident that

$$E[\alpha\tilde{u}_A^*(\theta,\eta) + (1-\alpha)\tilde{u}_S^*(\theta,\eta) \mid \eta] > E[\alpha u_A(\theta,\eta) + (1-\alpha)u_S(\theta,\eta) \mid \eta],$$

$$\tilde{u}_A^*(\theta,\eta) \geq u_A(\theta,\eta)$$

and

$$E[\tilde{u}_S^*(\theta,\eta) \mid \eta] \geq E[u_S(\theta,\eta) \mid \eta].$$

We can find $m^c(\theta,\eta) \in \Delta(M_A \times M_S)$ for GC such that

$$(X_A(m^c(\theta,\eta)) + X_S(m^c(\theta,\eta)), q(m^c(\theta,\eta))) = (\hat{X}(\tilde{m}^*(\theta \mid \eta)), \hat{q}(\tilde{m}^*(\theta \mid \eta))).$$

Now construct a new side-contract $SC(\eta)$ which realizes

$$(\tilde{u}_A^*(\theta,\eta), \tilde{u}_S^*(\theta,\eta), \hat{q}(\tilde{m}^*(\theta \mid \eta)))$$

as a PBE outcome of the game $GC \circ SC(\eta)$, contradicting the hypothesis that (u_A, u_S, q) is realized in a WPBE (wc), since by construction the interim participation constraints are satisfied. $SC(\eta)$ is specified as follows:

- $D^p \equiv D^{p*}$ where $D^{p*} = (D_A^{p*}, D_S^{p*})$ are A and S's message sets at the participation stage of the original equilibrium side-contract $SC^*(\eta)$.

- $D_A^e = \Theta(\eta)$ and $D_S^e = \{\phi\}$

- A's choice of $d_A^e = \theta \in \Theta(\eta)$ generates the report to P according to $m^c(\theta,\eta)$, associated with the transfers to A and P respectively:

  $$t_A(\theta,\eta) = \tilde{u}_A^*(\theta,\eta) - [X_A(m^c(\theta,\eta)) - \theta q(m^c(\theta,\eta))]$$

  and

  $$t_S(\theta,\eta) = \tilde{u}_S^*(\theta,\eta) - X_S(m^c(\theta,\eta)).$$

For this side-contract $SC(\eta)$, we claim the following is a PBE of the game $GC \circ SC(\eta)$. Given any (d_A^p, d_S^p) with $d_A^p \neq e_A$ and $d_S^p \neq e_S$ at the participation stage, A always selects $d_A^e = \theta$, since $\theta^{'} = \theta$ maximizes

$$X_A(m^c(\theta^{'}, \eta)) - \theta q(m^c(\theta^{'}, \eta)) + t_A(\theta^{'}, \eta) = \tilde{u}_A^*(\theta^{'}, \eta) + (\theta^{'} - \theta)\hat{q}(\tilde{m}^*(\theta \mid \eta)).$$

At the participation stage, A is indifferent among any $d_A^p \in D_A^p \backslash \{e_A\}$ as the optimal response to $d_S^p \neq e_S$, since the same outcome is realized in the continuation for any of these choices. Therefore it is optimal for A to choose the same $d_A^*(\theta, \eta)$ as in the original equilibrium. It implies that S's rejection induces non-cooperative play of GC based on the same updated beliefs as in the original equilibrium. $E[\tilde{u}_S^*(\theta, \eta) \mid \eta] \geq E[u_S(\theta, \eta) \mid \eta]$ guarantees S's participation. On the other hand, specify that A's choice of $d_A^p = e_A$ induces non-cooperative play of GC based on the same beliefs as in the original equilibrium. It guarantees A's participation $d_A^p \neq e_A$. Hence this is a PBE resulting in $(\tilde{u}_A^*(\theta, \eta), \tilde{u}_S^*(\theta, \eta))$, completing the argument. This completes the proof of the necessity.

Proof of Sufficiency

Take an allocation which is WCP(α) and satisfies the interim participation constraints. To show that it is implementable as a WPBE(wc) outcome, consider the grand contract GC:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \Pi \cup \{e_S\}$$

$$X_A(m_A, m_S) = X_S(m_A, m_S) = q(m_A, m_S) = 0$$

for (m_A, m_S) such that either $m_A = e_A$ or $m_S = e_S$.

- $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (u_A(\theta_A, \eta_S) + \theta_A q(\theta_A, \eta_S), q(\theta_A, \eta_S))$ for $\eta_A = \eta_S$ and $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (-T, 0)$ for $\eta_A \neq \eta_S$

- $X_S((\theta_A, \eta_A), \eta_S) = u_S(\theta_A, \eta_A)$ for $\eta_S = \eta_A$ and $X_S((\theta_A, \eta_A), \eta_S) = -T$ for $\eta_S \neq \eta_A$

where $T > 0$ is sufficiently large. The WCP(α) property implies that $u_A(\theta, \eta) \geq u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta)$. The interim participation constraints imply that this grand contract has a non-cooperative pure strategy equilibrium

$$(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$$

based on prior beliefs. For this grand contract, we claim there exists a WPBE(wc) resulting in $(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$. This requires us to check that there is no alternative $SC(\eta)$ in any state η with an associated PBE of the continuation game which generates a higher expected payoff for the third party, without making S or any type of A worse off. With sufficiently large $T > 0$, the third party never benefits from a side-contract which instructs the coalition to submit a report to P with $\eta_A \neq \eta_S$. Then the WCP(α) property implies that the third party does not benefit from any manipulation of the report to P, while guaranteeing $E[u_S(\theta, \eta) \mid \eta]$ to S and $u_A(\theta, \eta)$ to A. ■

Appendix C: Proofs of Results in the Text

Proof of Proposition 1: Consider the necessity part. Suppose the allocation (u_A, u_S, q) is WCP. Then the null side contract is optimal for S for every η, so must be feasible in $P(\eta)$. This implies $(u_A(\theta, \eta), q(\theta, \eta))$ satisfies A's incentive compatibility condition. Now consider the problem $P(\eta)$. The incentive constraint

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta^{'}, \eta) + (\theta^{'} - \theta)\hat{q}(\tilde{m}(\theta^{'} \mid \eta))$$

is equivalent to

$$\tilde{u}_A(\theta, \eta) = \tilde{u}_A(\bar{\theta}(\eta), \eta) + \int_{\theta}^{\bar{\theta}(\eta)} \hat{q}(\tilde{m}(y \mid \eta))dy$$

and $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ. Then the problem can be rewritten as

$$\max E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta]$$

subject to $\tilde{m}(\theta \mid \eta) \in \Delta(\hat{M})$ where $\hat{M} \equiv K \cup \{e\}$,

$$\tilde{u}_A(\theta, \eta) = \tilde{u}_A(\bar{\theta}(\eta), \eta) + \int_{\theta}^{\bar{\theta}(\eta)} \hat{q}(\tilde{m}(y \mid \eta))dy \geq u_A(\theta, \eta)$$

and $\hat{q}(\tilde{m}(\theta \mid \eta))$ non-increasing in θ. Since randomized side contracts can be chosen, the objective function is concave and the feasible set is convex. So the solution maximizes (subject to the constraint $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ) the following Lagrangian expression corresponding to some non-decreasing function $\tilde{\Lambda}(\theta \mid \eta)$:

$$\begin{aligned} \mathcal{L} &\equiv E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta)|\eta] \\ &+ \int_{[\underline{\theta}(\eta), \bar{\theta}(\eta)]} [\tilde{u}_A(\theta, \eta) - u_A(\theta, \eta)]d\tilde{\Lambda}(\theta \mid \eta) \end{aligned}$$

where $\hat{X}(\tilde{m}), \hat{q}(\tilde{m})$ denote expected values of $\hat{X}(m), \hat{q}(m)$ taken with respect to probability measure $\tilde{m}$ over $m \in \hat{M}$. Note that without loss of generality, $\tilde{u}_A(\theta, \eta)$ is a deterministic function.

A's incentive constraint implies $\tilde{u}_A(\theta, \eta)$ is continuous on $\Theta(\eta)$. Hence integration by parts yields:

$$\begin{aligned}
& \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} \tilde{u}_A(\theta,\eta) d\tilde{\Lambda}(\theta \mid \eta) = \tilde{\Lambda}(\bar{\theta}(\eta) \mid \eta)\tilde{u}_A(\bar{\theta}(\eta),\eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)\tilde{u}_A(\underline{\theta}(\eta),\eta) \\
+ & \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} \tilde{\Lambda}(\theta \mid \eta)\hat{q}(\tilde{m}(\theta \mid \eta))d\theta \\
= & [\tilde{\Lambda}(\bar{\theta}(\eta) \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)]\tilde{u}_A(\bar{\theta}(\eta),\eta) \\
+ & \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} [\tilde{\Lambda}(\theta \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)]\hat{q}(\tilde{m}(\theta \mid \eta))d\theta.
\end{aligned}$$

The second equality comes from

$$\tilde{u}_A(\underline{\theta}(\eta),\eta) = \tilde{u}_A(\bar{\theta}(\eta),\eta) + \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} \hat{q}(\tilde{m}(y \mid \eta))dy.$$

Next consider the effect of raising uniformly A's outside option function from $u_A(\theta, \eta)$ to $u_A(\theta, \eta) + \Delta$ where Δ is an arbitrary positive scalar. It is evident that the solution is unchanged, except that $\tilde{u}_A(\theta, \eta)$ is raised uniformly by Δ. Hence the maximized payoff of S must fall by Δ, implying that

$$\int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} \Delta d\tilde{\Lambda}(\theta \mid \eta) = [\tilde{\Lambda}(\bar{\theta}(\eta) \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)]\Delta = \Delta,$$

and so $\tilde{\Lambda}(\bar{\theta}(\eta) \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta) = 1$ in the optimal solution. Now define $\Lambda(\theta \mid \eta) \equiv \tilde{\Lambda}(\theta \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)$. Then $\Lambda(\theta \mid \eta)$ is non-decreasing in θ with $\Lambda(\underline{\theta}(\eta) \mid \eta) = 0$ and $\Lambda(\bar{\theta}(\eta) \mid \eta) = 1$.

This implies

$$\begin{aligned}
\mathcal{L} \equiv & \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} [\hat{X}(\tilde{m}(\theta \mid \eta)) - \pi(\theta \mid \eta)\hat{q}(\tilde{m}(\theta \mid \eta))]dF(\theta \mid \eta) \\
- & \int_{(\underline{\theta}(\eta),\bar{\theta}(\eta)]} u_A(\theta,\eta) d\Lambda(\theta \mid \eta) \qquad (15)
\end{aligned}$$

where $\pi(\theta \mid \eta) \equiv \theta + \frac{F(\theta|\eta) - \Lambda(\theta|\eta)}{f(\theta|\eta)}$. This has to be maximized subject to the constraint that $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ. This reduces to the unconstrained maximization of the corresponding expression where the CSC function $\pi(\cdot \mid \eta)$ is replaced by the corresponding CVC function $z(\cdot \mid \eta)$ using the ironing procedure relative to the cdf $F(\theta \mid \eta)$.

If $\tilde{m}^*(\theta \mid \eta)$ is optimal in problem $P(\eta)$, there exists $\pi(\cdot \mid \eta) \in Y(\eta)$ so that the optimal side contract $\tilde{m} = \tilde{m}^*(\theta \mid \eta)$ maximizes

$$\hat{X}(\tilde{m}(\theta \mid \eta)) - z(\theta \mid \eta)\hat{q}(\tilde{m}(\theta \mid \eta))$$

where $z(\theta \mid \eta) \equiv z(\theta \mid \pi(\cdot \mid \eta), \eta)$. Moreover $\hat{q}(\tilde{m}^*(\theta \mid \eta))$ must be non-increasing in θ and flat on any interval of θ which is a subset of $\Theta(\pi(\cdot \mid \eta), \eta)$.

If the optimal side contract is degenerate and concentrated at (θ, η), it must be the case that

$$\hat{X}(\theta, \eta) - z(\theta \mid \eta)\hat{q}(\theta, \eta) \geq \hat{X}(\tilde{m}^{'}) - z(\theta \mid \eta)\hat{q}(\tilde{m}^{'})$$

for any $\tilde{m}^{'} \in \Delta(\hat{M})$. This implies

$$\hat{X}(\theta, \eta) - z(\theta \mid \eta)q(\theta, \eta) \geq \hat{X}(\theta^{'}, \eta^{'}) - z(\theta \mid \eta)q(\theta^{'}, \eta^{'})$$

$$\hat{X}(\theta, \eta) - z(\theta \mid \eta)q(\theta, \eta) \geq 0$$

for any $(\theta, \eta), (\theta^{'}, \eta^{'})$, implying (i) in the proposition. Obviously $q(\theta, \eta)$ must be non-increasing in θ and must be flat on any interval of θ which is a subset of $\Theta(\pi(\cdot \mid \eta), \eta)$ (implying (iii) in the proposition).

Now consider the sufficiency part. Consider any state η. Suppose there is a CSC function $\pi(\cdot \mid \eta) \in Y(\eta)$ which is ironed to yield the CVC function $z(\cdot|\eta)$ such that $(u_S(\theta, \eta), u_A(\theta, \eta), q(\theta, \eta))$ satisfies all the conditions in the proposition. Define $(\hat{X}(m), \hat{q}(m))$ on $\hat{M} \equiv K \cup \{e\}$ such that

$$(\hat{X}(\theta, \eta), \hat{q}(\theta, \eta)) = (u_S(\theta, \eta) + u_A(\theta, \eta) + \theta q(\theta, \eta), q(\theta, \eta))$$

and

$$(\hat{X}(e), \hat{q}(e)) = (0, 0).$$

and extend this to $(\hat{X}(\tilde{m}), \hat{q}(\tilde{m}))$ on $\Delta(\hat{M})$ in the obvious manner. Consider the problem $P(\eta)$ as selection of $\tilde{m}(\theta|\eta), \tilde{u}_A(\theta, \eta)$ to maximize

$$E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta]$$

subject to

$$\tilde{u}_A(\theta, \eta) \geq u_A(\theta, \eta)$$

for any $\theta \in \Theta(\eta)$,

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}(\tilde{m}(\theta' \mid \eta))$$

for any $\theta, \theta' \in \Theta(\eta)$. For $\tilde{u}_A(\theta, \eta)$ which satisfies constraints of the problem, we have

$$\int_{[\underline{\theta}(\eta), \bar{\theta}(\eta)]} [\tilde{u}_A(\theta, \eta) - u_A(\theta, \eta)] d\Lambda(\theta \mid \eta) \geq 0.$$

Then

$$\begin{aligned}
& E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta] \\
\leq\ & E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta] \\
+\ & \int_{[\underline{\theta}(\eta), \bar{\theta}(\eta)]} [\tilde{u}_A(\theta, \eta) - u_A(\theta, \eta)] d\Lambda(\theta \mid \eta).
\end{aligned}$$

Now consider the problem of maximizing the right hand side of this inequality, subject to the constraint that $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ. Using the same steps in the proof of the necessity part, this can be expressed as a problem of selecting $\tilde{m}(\theta|\eta)$ to maximize the Lagrangean (15) subject to the constraint that $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ. Conditions (i)-(iii) imply that the right-hand-side is maximized at $\tilde{m}(\theta \mid \eta) = (\theta, \eta)$ and $\tilde{u}_A(\theta, \eta) = u_A(\theta, \eta)$. Since

$$\int_{[\underline{\theta}(\eta), \bar{\theta}(\eta)]} [\tilde{u}_A(\theta, \eta) - u_A(\theta, \eta)] d\Lambda(\theta \mid \eta) = 0$$

when $\tilde{u}_A(\theta, \eta) = u_A(\theta, \eta)$, this shows that the left hand side of the above inequality is also maximized at $\tilde{m}(\theta \mid \eta) = (\theta, \eta)$ and $\tilde{u}_A(\theta, \eta) = u_A(\theta, \eta)$. Hence $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta, \eta)) = ((\theta, \eta), u_A(\theta, \eta))$ solves $P(\eta)$. ■

Proof of Proposition 2:

Proof of Necessity

Suppose (u_A, u_S, q) is implementable in the weak collusion game. It is evident that it satisfies interim participation constraints of A and S. Here we show that it is also a WCP allocation. Suppose not. Then there exists $\eta \in \Pi$ such that $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta, \eta)) = ((\theta, \eta), u_A(\theta, \eta))$ does not solve the side-contracting problem $P(\eta)$. Suppose that $(\tilde{m}^*(\theta \mid \eta), \tilde{u}_A^*(\theta, \eta))$ is the solution of $P(\eta)$. Defining

$$\tilde{u}_S^*(\theta, \eta) \equiv \hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta \hat{q}(\tilde{m}^*(\theta \mid \eta)) - \tilde{u}_A^*(\theta, \eta),$$

we have

$$E[\tilde{u}_S^*(\theta, \eta) \mid \eta] > E[u_S(\theta, \eta) \mid \eta]$$

and

$$\tilde{u}_A^*(\theta, \eta) \geq u_A(\theta, \eta)$$

for any $\theta \in \Theta(\eta)$. Since (u_A, u_S, q) is implementable in the weak collusion game, there exists a grand contract GC and an associated WPBE(wc) which results in this allocation. From the property of WPBE(wc), there exists belief $p(\eta)$ and non-cooperative equilibrium $c(\eta)$ of GC based on the belief $p(\eta)$ such that A's payoff is not better than $u_A(\theta, \eta)$ for any $\theta \in \Theta(\eta)$.

For $\tilde{m}^*(\theta \mid \eta) \in \Delta(K \cup e)$, there exists $\tilde{m}^c(\theta, \eta) \in \Delta(M_A \times M_S)$ such that

$$(\hat{X}(\tilde{m}^*(\theta \mid \eta)), \hat{q}(\tilde{m}^*(\theta \mid \eta))) = (X_A(\tilde{m}^c(\theta, \eta)) + X_S(\tilde{m}^c(\theta, \eta)), q(\tilde{m}^c(\theta, \eta))).$$

Given GC and η, suppose that S offers the side-contract $SC^c(\eta)$ such that the report to P is selected according to $\tilde{m}^c(\theta', \eta)$ on the basis of A's report of $\theta' \in \Theta(\eta)$, associated with the transfer to A:

$$t_A^c(\theta', \eta) = \tilde{u}_A^*(\theta', \eta) - [X_A(\tilde{m}^c(\theta', \eta)) - \theta' q(\tilde{m}^c(\theta', \eta)))].$$

Now construct a Perfect Bayesian Equilibrium (PBE) in the game induced by GC and $SC^c(\eta)$, as follows. It is evident that if A accepts this side-contract, it is optimal for him to truthfully report $\theta \in \Theta(\eta)$, generating payoffs $\tilde{u}_A^*(\theta, \eta)$ and $\tilde{u}_S^*(\theta, \eta)$ for A and S respectively. If A rejects the side-contract, A and S play $c(\eta)$ based on the belief $p(\eta)$ specified above. Since $\tilde{u}_A^*(\theta,\eta) \geq u_A(\theta,\eta)$, all types of A participate in the side-contract, given this choice of non-cooperative equilibrium in the event that A rejects the side-contract. The argument shows that $(\tilde{u}_A^*(\theta,\eta), \tilde{u}_S^*(\theta,\eta))$ is realized as a PBE outcome. Since S is better off without making any type of A worse off, it contradicts the fact that (u_A, u_S, q) is realized as the outcome of a WPBE(wc).

Proof of Sufficiency

Suppose that (u_A, u_S, q) is a WCP allocation satisfying interim participation constraints of A and S. We show that there exists a grand contract which realizes (u_A, u_S, q) as a WPBE(wc) outcome. Consider the following grand contract, corresponding to $T > 0$ chosen sufficiently large:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \Pi \cup \{e_S\}$$

$$X_A(m_A, m_S) = X_S(m_A, m_S) = q(m_A, m_S) = 0$$

for (m_A, m_S) such that either $m_A = e_A$ or $m_S = e_S$.

- $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (u_A(\theta_A, \eta_S) + \theta_A q(\theta_A, \eta_S), q(\theta_A, \eta_S))$ for $\eta_A = \eta_S$ and $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (-T, 0)$ for $\eta_A \neq \eta_S$

- $X_S((\theta_A, \eta_A), \eta_S) = u_S(\theta_A, \eta_A)$ for $\eta_S = \eta_A$ and $X_S((\theta_A, \eta_A), \eta_S) = -T$ for $\eta_S \neq \eta_A$

WCP implies that $u_A(\theta,\eta) \geq u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta)$ for any $\theta, \theta' \in \Theta(\eta)$. Then together with interim participation constraints of A and S, this grand contract has a non-cooperative truthful equilibrium $(m_A^*(\theta,\eta), m_S^*(\eta)) = ((\theta,\eta),\eta)$ based on prior beliefs. Then there exists a WPBE where S offers the null side-contract on the equilibrium path for any $\eta \in \Pi$. In this WPBE, for any non-null side-contract, A's rejection always induces the truthful equilibrium based on prior beliefs. Then since (u_A, u_S, q) is WCP, S cannot benefit from any non-null side-contract. This equilibrium also satisfies the robustness criterion in WPBE(wc), since there is no room for S to achieve a higher payoff, while leaving a payoff of at least $u_A(\theta, \eta)$ to all types of A. Therefore (u_A, u_S, q) is a WPBE(wc) outcome, given GC. ∎

Proof of Proposition 3:

At the first step, note that the optimal side contract problem for S in DS involves an outside option for A which is identically zero. This reduces to a standard problem of contracting with a single agent with adverse selection and an outside option of zero, where the principal has a prior distribution $F(\theta|\eta)$ over the agent's cost θ in state η. The CSC function equals $h(\theta|\eta)$, and the CVC function $z(\theta|\eta)$ reduces to $\hat{h}(\theta|\eta)$ obtained by applying the ironing rule to $h(\theta|\eta)$ and distribution $F(\theta|\eta)$.

Given this, P's contract with S in DS is effectively a contracting problem for P with a single supplier whose unit supply cost is $\hat{h}(\theta|\eta)$. P's prior over this supplier's cost is given by distribution function

$$G(h) \equiv \Pr((\theta, \eta) \mid \hat{h}(\theta \mid \eta) \leq h)$$

for $h \geq \underline{\theta}$ and $G(h) = 0$ for $h < \underline{\theta}$. Let $G(h \mid \eta)$ denote the cumulative distribution function of $h = \hat{h}(\theta \mid \eta)$ conditional on η:

$$G(h \mid \eta) \equiv \Pr(\theta \mid \hat{h}(\theta \mid \eta) \leq h, \eta)$$

for $h \geq \hat{h}(\underline{\theta}(\eta) \mid \eta)(= \underline{\theta}(\eta))$ and $G(h \mid \eta) = 0$ for $h < \underline{\theta}(\eta)$. Then $G(h) = \Sigma_{\eta \in \Pi} p(\eta) G(h \mid \eta)$. Since $\hat{h}(\theta \mid \eta)$ is continuous on $\Theta(\eta)$, $G(h \mid \eta)$ is strictly increasing in h on $[\underline{\theta}(\eta), \hat{h}(\bar{\theta}(\eta) \mid \eta)]$. However, $G(h \mid \eta)$ may fail to be left-continuous.

Hence P's problem in DS reduces to

$$\max E_h[V(q(h)) - X(h)]$$

subject to

$$X(h) - hq(h) \geq X(h^{'}) - hq(h^{'})$$

for any $h, h^{'} \in [\underline{\theta}, \bar{h}]$ and

$$X(h) - hq(h) \geq 0$$

for any $h \in [\underline{\theta}, \bar{h}]$ where the distribution function of h is $G(h)$ and $\bar{h} \equiv \max_{\eta \in \Pi} \hat{h}(\bar{\theta}(\eta) \mid \eta)$. The corresponding problem in NS is

$$\max E_{\theta}[V(q(\theta)) - X(\theta)]$$

subject to

$$X(\theta) - \theta q(\theta) \geq X(\theta^{'}) - \theta q(\theta^{'})$$

for any $\theta, \theta^{'} \in \Theta$ and

$$X(\theta) - \theta q(\theta) \geq 0$$

for any $\theta \in \Theta$. The two problems differ only in the underlying cost distributions of P: $G(h)$ in the case of DS and $F(\theta)$ in the case of NS. Since $\theta < \hat{h}(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$,

$$G(h \mid \eta) \equiv \Pr(\theta \mid \hat{h}(\theta \mid \eta) \leq h, \eta) < \Pr(\theta \mid \theta \leq h, \eta) = F(h \mid \eta)$$

for $h \in (\underline{\theta}(\eta), \hat{h}(\bar{\theta}(\eta) \mid \eta))$, implying

$$G(h) = \Sigma_{\eta \in \Pi} p(\eta) G(h \mid \eta) < \Sigma_{\eta \in \Pi} p(\eta) F(h \mid \eta) = F(h)$$

for any $h \in (\underline{\theta}, \bar{h})$. Therefore the distribution of h in DS (strictly) dominates that of θ in NS in the first order stochastic sense.

It remains to show that this implies that P must earn a lower profit in DS. We prove the following general statement. Consider two contracting problems with a single supplier which differ only in regard to the cost distributions G_1 and G_2, where $G_1(h) < G_2(h)$ for any $h \in (\underline{h}, \bar{h})$. Let the maximized profit of P with distribution G be denoted $W(G)$. We will show $W(G_1) < W(G_2)$.

Let $q_1(h)$ denote the optimal solution of the problem based on $G_1(h)$.

(i) First we show that $V^{'}(q_1(h)) < h$ does not hold for any h. Suppose otherwise that there exists some interval over which $V^{'}(q_1(h)) < h$. Then we can replace the portion of $q_1(h)$ with $V^{'}(q_1(h)) < h$ by $q^*(h)$ with $V^{'}(q^*(h)) = h$, without violating the constraint that $q(h)$ is non-increasing. It raises the value of the objective function, since $V(q_1(h)) - hq_1(h) < V(q_1^*(h)) - hq_1^*(h)$ for h where $q_1(h)$ is replaced by $q^*(h)$, and $\int_h^{\bar{h}} q(y)dy$ decreases with this replacement. This is a contradiction.

(ii) Next we show that for any $h^{'} \in [\underline{h}, \bar{h})$, there exists a subinterval of $[h^{'}, \bar{h})$ over which $V^{'}(q_1(h)) > h$. Otherwise, there exists $h^{'} \in [\underline{h}, \bar{h})$ such that $q_1(h) = q^*(h)$ almost everywhere on $[h^{'}, \bar{h})$. Then for any $h \in [h^{'}, \bar{h})$,

$$V(q^*(h)) - hq^*(h) - \int_h^{\bar{h}} q^*(y)dy = V(q^*(\bar{h})) - \bar{h}q^*(\bar{h}),$$

since $V(q^*(h)) - hq^*(h) = \int_h^{\bar{h}} q^*(y)dy + V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})$ (which follows from the Envelope Theorem: $d[V(q^*(h)) - hq^*(h)]/dh = -q^*(h)$). Then

$$\begin{aligned} & W(G_1) = (1 - G_1(h^{'}))[V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})] \\ + \; & G_1(h^{'})E[V(q_1(h)) - hq_1(h) - \int_h^{h^{'}} q_1(y)dy \mid h \leq h^{'}] - G_1(h^{'}) \int_{h^{'}}^{\bar{h}} q^*(y)dy. \end{aligned}$$

Now consider output schedule $q(h)$ such that $q(h) = q_1(h)$ for $h \leq h^{'}$ and $q(h) = q^*(\bar{h})$ for $h > h^{'}$. It is evident that $q(h)$ is non-increasing in h and generates a higher value of the objective function, since $\int_{h'}^{\bar{h}} q^*(y)dy > \int_{h'}^{\bar{h}} q^*(\bar{h})dy$. This is a contradiction.

(iii) We show there does not exist q such that $q_1(h) = q$ almost everywhere. Otherwise, $q_1(h) = q$ almost everywhere for some q. Then

$$V(q) - hq - \int_h^{\bar{h}} qdy = V(q) - \bar{h}q,$$

which is not larger than $V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})$ which equals $\max_{\tilde{q}}[V(\tilde{q}) - \bar{h}\tilde{q}]$. We can show that the value of the objective function is increased by choosing the following output schedule $\tilde{q}(h)$:

$$\tilde{q}(h) = \begin{cases} q^*(\bar{h}) & h \in [h^*, \bar{h}] \\ q^*(\bar{h}) + \epsilon & h \in [\underline{h}, h^*] \end{cases}$$

where h^* is any element of $(\underline{h}, \bar{h})$, and $\epsilon > 0$ is chosen so that $V(q^*(\bar{h})+\epsilon)-V(q^*(\bar{h})) > \epsilon h^*$. This is possible since $\lim_{\epsilon\to 0} \frac{V(q^*(\bar{h})+\epsilon)-V(q^*(\bar{h}))}{\epsilon} = V^{'}(q^*(\bar{h})) = \bar{h}$, implying existence of $\epsilon > 0$ such that $V(q^*(\bar{h}) + \epsilon) - V(q^*(\bar{h})) > \epsilon h^*$ for any $h^* < \bar{h}$.

Then we obtain a contradiction, since

$$\begin{aligned} & V(q^*(\bar{h})) - \bar{h}q^*(\bar{h}) \\ < \; & (1 - G_1(h^*))[V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})] + G_1(h^*)[V(q^*(\bar{h}) + \epsilon) - \bar{h}q^*(\bar{h}) - \epsilon h^*] \\ = \; & \int_{\underline{h}}^{\bar{h}} [V(\tilde{q}(h)) - h\tilde{q}(h) - \int_h^{\bar{h}} \tilde{q}(y)dy]dG_1(h). \end{aligned}$$

(iv) Define

$$\Phi(h) \equiv V(q_1(h)) - hq_1(h) - \int_h^{\bar{h}} q_1(y)dy.$$

We claim that $\Phi(h)$ is left-continuous and bounded. First we show that $q_1(h)$ is left-continuous. Otherwise, there exists $h^{'} \in (\underline{h}, \bar{h})$ such that $q_1(h^{'}-) > q_1(h^{'})$. Now consider $\tilde{q}_1(h)$ (which is left-continuous at $h^{'}$) such that $\tilde{q}_1(h^{'}) = q_1(h^{'}-)$ and $\tilde{q}_1(h) = q_1(h)$ for any $h \neq h^{'}$. Defining $\tilde{\Phi}(h) \equiv V(\tilde{q}_1(h)) - h\tilde{q}_1(h) - \int_h^{\bar{h}} \tilde{q}_1(y)dy$, observe that $\tilde{\Phi}(h) = \Phi(h)$ for $h \neq h^{'}$ and $\tilde{\Phi}(h) > \Phi(h)$ when $h = h^{'}$. Then

$$\begin{aligned} \int_{[\underline{h},\bar{h}]} \tilde{\Phi}(h)dG(h) &= \int_{[\underline{h},\bar{h}]\setminus h'} \tilde{\Phi}(h)dG(h) + \tilde{\Phi}(h^{'})[G(h^{'}+) - G(h^{'}-)] \\ \geq \; & \int_{[\underline{h},\bar{h}]\setminus h'} \tilde{\Phi}(h)dG(h) + \Phi(h^{'})[G(h^{'}+) - G(h^{'}-)] = \int_{[\underline{h},\bar{h}]} \Phi(h)dG(h) \end{aligned}$$

with strict inequality if $G(h)$ is discontinuous at $h = h^{'}$. This is a contradiction. This implies in turn that $\Phi(h)$ is also left-continuous. Moreover, $\Phi(h)$ is bounded, since

$$\Phi(h) \leq \Phi(\underline{h}) \leq V(q_1(\underline{h})) - \underline{h}q_1(\underline{h}) \leq V(q^*(\underline{h})) - \underline{h}q^*(\underline{h}) < \infty$$

because of $\underline{h} > 0$, and

$$\Phi(h) \geq \Phi(\bar{h}) = V(q_1(\bar{h})) - \bar{h}q_1(\bar{h}) \geq 0$$

because of $V^{'}(q) > V^{'}(q_1(\bar{h})) \geq \bar{h}$ for $q < q_1(\bar{h})$ and $V(0) = 0$.

(v) We claim that $\Phi(h)$ is non-increasing in h and is not constant on $(\underline{h}, \bar{h})$. To show the former, note that for any h, we have

$$\begin{aligned}
& \lim_{\epsilon \to 0+} \frac{\Phi(h+\epsilon) - \Phi(h)}{\epsilon} \\
= & \lim_{\epsilon \to 0+} (1/\epsilon)[V(q_1(h+\epsilon)) - (h+\epsilon)q_1(h+\epsilon) - \int_{h+\epsilon}^{\bar{h}} q_1(y)dy \\
- & [V(q_1(h)) - hq_1(h) - \int_h^{\bar{h}} q_1(y)dy]] \\
= & [V^{'}(\hat{q}(h)) - h] \lim_{\epsilon \to 0+} \frac{q_1(h+\epsilon) - q_1(h)}{\epsilon} \\
- & q_1(h+) + \lim_{\epsilon \to 0+} (1/\epsilon) \int_h^{h+\epsilon} q_1(y)dy \\
= & [V^{'}(\hat{q}(h)) - h] \lim_{\epsilon \to 0+} \frac{q_1(h+\epsilon) - q_1(h)}{\epsilon}
\end{aligned}$$

for some $\hat{q}(h) \in [q_1(h+), q_1(h)]$. This is non-positive since $V^{'}(\hat{q}(h)) \leq V^{'}(q_1(h+)) \leq h$ and $\lim_{\epsilon \to 0+} \frac{q_1(h+\epsilon) - q_1(h)}{\epsilon} \leq 0$. Because of left-continuity of $\Phi(h)$, it implies that $\Phi(h)$ is non-increasing in h.

Next we show that $\Phi(h)$ is not constant on $(\underline{h}, \bar{h})$. First we consider the case that there exists $h \in (\underline{h}, \bar{h})$ such that $q_1(h+) < q_1(h-)$. Then

$$\begin{aligned}
& \Phi(h+) \\
= & V(q_1(h+)) - hq_1(h+) - \int_h^{\bar{h}} q_1(y)dy] \\
< & V(q_1(h-)) - hq_1(h-) - \int_h^{\bar{h}} q_1(y)dy = \Phi(h-)
\end{aligned}$$

The inequality follows from $V^{'}(q_1(h+)) > V^{'}(q_1(h-)) \geq V^{'}(q^*(h)) = h$. Therefore $\Phi(h)$ decreases discontinuously at h, implying that $\Phi(h)$ is not constant on $(\underline{h}, \bar{h})$. Second we consider the case that $q(h)$ is continuous on $(\underline{h}, \bar{h})$. Then from (ii) and (iii) above, there exists an interval (h^-, h^+) with the positive measure such that $q_1(h)$ is strictly decreasing and $V^{'}(q_1(h)) > h$ on (h^-, h^+). $\Phi(h)$ is continuous and almost everywhere differentiable (because of monotonicity of $q_1(h)$). At any point of differentiability,

$$\Phi^{'}(h) = [V^{'}(q_1(h)) - h]q_1^{'}(h).$$

This is negative almost everywhere on (h^-, h^+). Hence $\Phi(h)$ is strictly decreasing in h on (h^-, h^+).

(vi) Now consider the contracting problem with cost distribution $G_2(h)$. Since $q_1(h)$ is non-increasing in h, it is feasible for P to select this output schedule when the cost distribution is G_2. Hence $W(G_2) \geq \int_{\underline{h}}^{\bar{h}} \Phi(h) dG_2(h)$. Therefore if $\int_{\underline{h}}^{\bar{h}} \Phi(h) dG_2(h) > \int_{\underline{h}}^{\bar{h}} \Phi(h) dG_1(h) = W(G_1)$, it follows that $W(G_2) > W(G_1)$. Since $G_1(h)$ is right-continuous and $\Phi(h)$ is left-continuous and bounded, we can integrate by parts:

$$\int_{\underline{h}}^{\bar{h}} \Phi(h) dG_1(h) + \int_{\underline{h}}^{\bar{h}} G_1(h) d\Phi(h) = \Phi(\bar{h}) G_1(\bar{h}) - \Phi(\underline{h}) G_1(\underline{h}) = \Phi(\bar{h}).$$

Similarly for $G_2(h)$,

$$\int_{\underline{h}}^{\bar{h}} \Phi(h) dG_2(h) + \int_{\underline{h}}^{\bar{h}} G_2(h) d\Phi(h) = \Phi(\bar{h}) G_2(\bar{h}) - \Phi(\underline{h}) G_2(\underline{h}) = \Phi(\bar{h}).$$

Hence

$$\int_{\underline{h}}^{\bar{h}} \Phi(h) dG_2(h) - \int_{\underline{h}}^{\bar{h}} \Phi(h) dG_1(h) = \int_{\underline{h}}^{\bar{h}} [G_1(h) - G_2(h)] d\Phi(h).$$

By (iv) and $G_2(h) > G_1(h)$ for $h \in (\underline{h}, \bar{h})$, this is positive. ■

Proof of Proposition 4:

Step 1: For any $\eta \in \Pi$ and any closed interval $[\theta', \theta''] \subset \Theta(\eta)$ such that $\underline{\theta}(\eta) < \theta' < \theta'' < \bar{\theta}(\eta)$, there exists $\delta > 0$ such that $z(\cdot) \in Z(\eta)$ for any $z(\cdot)$ satisfying the following properties:

(i) $z(\theta)$ is increasing and differentiable with $|z(\theta) - \theta| < \delta$ and $|z'(\theta) - 1| < \delta$ for any $\theta \in \Theta(\eta)$

(ii) $z(\theta) = \theta$ for any $\theta \notin [\theta', \theta'']$.

Proof of Step 1

For arbitrary $\eta \in \Pi$ and arbitrary closed interval $[\theta', \theta''] \subset \Theta(\eta)$ such that $\underline{\theta}(\eta) < \theta' < \theta'' < \bar{\theta}(\eta)$, we choose ϵ_1 and ϵ_2 such that

$$\epsilon_1 \equiv \min_{\theta \in [\theta', \theta'']} f(\theta \mid \eta)$$

and

$$\epsilon_2 \equiv \max_{\theta \in [\theta', \theta'']} |f'(\theta \mid \eta)|.$$

From our assumptions that $f(\theta \mid \eta)$ is continuously differentiable and positive on $\Theta(\eta)$, $\epsilon_1 > 0$, and ϵ_2 is positive and bounded above. We choose $\delta > 0$ such that

$$\delta \in (0, \frac{\epsilon_1}{\epsilon_1 + \epsilon_2}).$$

For this δ, it is obvious that there exists $z(\theta)$ which satisfies conditions (i) and (ii) of the statement. Define

$$\Lambda(\theta \mid \eta) \equiv (\theta - z(\theta)) f(\theta \mid \eta) + F(\theta \mid \eta).$$

Since $z(\theta)$ is differentiable on $\Theta(\eta)$, $\Lambda(\theta \mid \eta)$ is also so. It is equal to $\Lambda(\theta \mid \eta) = F(\theta \mid \eta)$ on $\theta \notin [\theta', \theta'']$. For $\theta \in [\theta', \theta'']$,

$$\begin{aligned} \frac{\partial \Lambda(\theta \mid \eta)}{\partial \theta} &= (2 - z'(\theta)) f(\theta \mid \eta) + (\theta - z(\theta)) f'(\theta \mid \eta) > (1 - \delta) f(\theta \mid \eta) - \delta |f'(\theta \mid \eta)| \\ &\geq (1 - \delta)\epsilon_1 - \delta \epsilon_2. \end{aligned}$$

This is positive by the definition of $(\epsilon_1, \epsilon_2, \delta)$. Then $\Lambda(\theta \mid \eta)$ is increasing in θ on $\Theta(\eta)$ with $\Lambda(\underline{\theta}(\eta) \mid \eta) = 0$ and $\Lambda(\bar{\theta}(\eta) \mid \eta) = 1$. Since $z(\theta)$ is increasing in θ by the definition, it is preserved even by ironing rule. Therefore $z(\cdot) \in Z(\eta)$. ■

Step 2: There exist $\eta \in \Pi$ and an interval of θ with positive measure such that $\frac{F(\theta|\eta)}{f(\theta|\eta)} / \frac{F(\theta)}{f(\theta)}$ is increasing in θ.

The proof of Step 2

Define

$$A(\theta \mid \eta) \equiv \frac{F(\theta \mid \eta)}{f(\theta \mid \eta)} / \frac{F(\theta)}{f(\theta)} \equiv \frac{\int_{\underline{\theta}(\eta)}^{\theta} f(y) a(\eta|y) dy}{a(\eta|\theta) F(\theta)}.$$

If the result is false, $A(\theta \mid \eta)$ is non-increasing in $\theta \in (\underline{\theta}(\eta), \bar{\theta}(\eta))$ for all η. Then

$$\begin{aligned} \partial A(\theta \mid \eta)/\partial \theta &= \frac{1}{F(\theta)^2 a(\eta \mid \theta)^2} [F(\theta) a(\eta \mid \theta)^2 f(\theta) \\ &- \int_{\underline{\theta}(\eta)}^{\theta} f(y) a(\eta \mid y) dy \{F(\theta) \partial a(\eta \mid \theta)/\partial \theta + f(\theta) a(\eta \mid \theta)\}] \leq 0 \end{aligned}$$

holds for $\theta \in (\underline{\theta}(\eta), \bar{\theta}(\eta))$. Equivalently

$$\partial a(\eta \mid \theta)/\partial\theta \geq \frac{f(\theta)}{F(\theta)}[1/A(\theta \mid \eta) - 1]a(\eta \mid \theta).$$

Define $\Pi(\theta) \equiv \{\eta \in \Pi \mid \theta \in (\underline{\theta}(\eta), \bar{\theta}(\eta))\}$. By $\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta) = 1$, $\Sigma_{\eta\in\Pi(\theta)} \partial a(\eta \mid \theta)/\partial\theta = 0$. This implies that

$$0 = \Sigma_{\eta\in\Pi(\theta)} \partial a(\eta \mid \theta)/\partial\theta \geq \frac{f(\theta)}{F(\theta)}[\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta)/A(\theta \mid \eta) - 1],$$

or $\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta)/A(\theta \mid \eta) \leq 1$ holds any for $\theta \in (\underline{\theta}, \bar{\theta})$. Since $1/A$ is convex in A and $\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta) A(\theta \mid \eta) = 1$,

$$\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta)/A(\theta \mid \eta) \geq 1/[\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta) A(\theta \mid \eta)] = 1$$

with strict inequality if there exists $\eta \in \Pi(\theta)$ such that $A(\theta \mid \eta) \neq 1$. This means that $A(\theta \mid \eta) = 1$ must hold for any $\eta \in \Pi(\theta)$ and any $\theta \in \Theta$. Then $h(\theta \mid \eta) = H(\theta)$ for any $(\theta, \eta) \in K$. This is a contradiction, since η is informative about θ. ■

Step 3:

From Step 2, we can choose $\eta^* \in \Pi$ and a closed interval $[\theta', \theta''] \subset \Theta(\eta^*)$ such that $\underline{\theta}(\eta^*) < \theta' < \theta'' < \bar{\theta}(\eta^*)$ and $A(\theta \mid \eta^*) \equiv \frac{F(\theta|\eta^*)}{f(\theta|\eta^*)} / \frac{F(\theta)}{f(\theta)}$ is increasing in θ on $[\theta', \theta'']$. According to the procedure in Step 1, we select $\delta > 0$ for η^* and $[\theta', \theta'']$. Then we also choose $\lambda > 0$, closed intervals $\Theta^L \subset [\theta', \theta'']$ and $\Theta^H \subset [\theta', \theta'']$,

$$\lambda < \frac{F(\theta)}{f(\theta)} / \frac{F(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} \text{ for } \theta \in \Theta^L \equiv [\underline{\theta}^L, \bar{\theta}^L] \subset [\theta', \theta'']$$

$$\lambda > \frac{F(\theta)}{f(\theta)} / \frac{F(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} \text{ for } \theta \in \Theta^H \equiv [\underline{\theta}^H, \bar{\theta}^H] \subset [\theta', \theta'']$$

with $\bar{\theta}^L < \underline{\theta}^H$. These conditions are equivalent to

$$H(\theta) - (1 - \lambda)\theta - \lambda h(\theta \mid \eta^*) > 0 \text{ for } \theta \in \Theta^L$$

and

$$H(\theta) - (1 - \lambda)\theta - \lambda h(\theta \mid \eta^*) < 0 \text{ for } \theta \in \Theta^H.$$

Step 4: Construction of $z(\theta \mid \eta)$

Now let us construct $z(\theta \mid \eta)$ which satisfies the following conditions.

(A) For $\eta \neq \eta^*$, $z(\theta \mid \eta) = \theta$ for any $\theta \in \Theta(\eta)$.

(B) For η^*, $z(\theta \mid \eta^*)$ satisfies

- (i) $z(\theta \mid \eta^*)$ is increasing and differentiable with $|z(\theta \mid \eta^*) - \theta| < \delta$ and $|z^{'}(\theta \mid \eta^*) - 1| < \delta$ for any $\theta \in \Theta(\eta^*)$
- (ii) $z(\theta \mid \eta^*) = \theta$ for any $\theta \notin \Theta^H \cup \Theta^L$
- (iii) For $\theta \in \Theta^L$, $z(\theta \mid \eta^*)$ satisfies (a) $z(\theta \mid \eta^*) \leq \theta$ with strict inequality for some subinterval of Θ^L of positive measure, and (b) $H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^*) > 0$ for any $z \in [z(\theta \mid \eta^*), \theta]$.
- (iv) For $\theta \in \Theta^H$, $z(\theta \mid \eta^*)$ satisfies (a) $z(\theta \mid \eta^*) \geq \theta$ with strict inequality for some some subinterval of Θ^H of positive measure, (b) $z(\theta \mid \eta^*) < h(\theta \mid \eta^*)$ and (c) $H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^*) < 0$ for any $z \in [\theta, z(\theta \mid \eta^*)]$.
- (v) $E[(z(\theta \mid \eta^*) - h(\theta \mid \eta^*))q^{NS}(z(\theta \mid \eta^*)) + \int_{z(\theta|\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*] = 0$.

We now argue there exists $z^*(\theta \mid \eta^*)$ which satisfies (B(i)-(v)). Step 3 guarantees that we can select $z(\theta \mid \eta^*)$ which satisfies (B(i)-(iv)). Since

$$(z - h(\theta \mid \eta^*))q^{NS}(z) + \int_z^{\bar{\theta}(\eta^*)} q^{NS}(y)dy$$

is increasing in z for $z < h(\theta \mid \eta^*)$, and

$$E[(\theta - h(\theta \mid \eta^*))q^{NS}(\theta) + \int_\theta^{\bar{\theta}(\eta^*)} q^{NS}(y)dy \mid \eta^*] = 0,$$

the choice of $z(\theta \mid \eta^*) \leq \theta$ on Θ_L (or $z(\theta \mid \eta^*) \geq \theta$ on Θ_H) reduces (or raises)

$$E[(z(\theta \mid \eta^*) - h(\theta \mid \eta^*))q^{NS}(z(\theta \mid \eta^*)) + \int_{z(\theta|\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*]$$

away from zero. For any pair of parameters α_H, α_L lying in $[0,1]$, define a function $z_{\alpha_L,\alpha_H}(\theta|\eta^*)$ which equals $(1-\alpha_L)z(\theta|\eta^*)+\alpha_L\theta$ on Θ_L, equals $(1-\alpha_H)z(\theta|\eta^*)+\alpha_H\theta$ on Θ_H and equals θ elsewhere. It is easily checked that any such function also satisfies conditions (B(i)–(iv)). Define

$$Q(\alpha_L,\alpha_U) \equiv E[(z_{\alpha_L,\alpha_H}(\theta \mid \eta^*)-h(\theta \mid \eta^*))q^{NS}(z_{\alpha_L,\alpha_H}(\theta \mid \eta^*))+\int_{z_{\alpha_L,\alpha_H}(\theta|\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*].$$

Then Q is continuously differentiable, strictly increasing in α_L and strictly decreasing in α_H. By (B(v)), $Q(1,1)=0$. The Implicit Function Theorem ensures existence of α_L^*, α_H^* both smaller than 1 such that $Q(\alpha_L^*,\alpha_H^*)=0$. Hence the function $z_{\alpha_L^*,\alpha_H^*}(\theta|\eta^*)$ satisfies (B(i)-(v)).

Step 5

By Step 1, $z(\cdot \mid \eta)$ constructed in Step 4 is in $Z(\eta)$ for any $\eta \in \Pi$. Consider the following allocation (u_A,u_S,q):

$$q(\theta,\eta) = q^{NS}(z(\theta \mid \eta))$$

$$u_A(\theta,\eta) = \int_{\theta}^{\bar{\theta}} q^{NS}(z(y \mid \eta))dy$$

$$u_S(\theta,\eta) = X^{NS}(z(\theta \mid \eta)) - \theta q^{NS}(z(\theta \mid \eta)) - \int_{\theta}^{\bar{\theta}(\eta)} q^{NS}(z(y \mid \eta))dy - \int_{\bar{\theta}(\eta)}^{\bar{\theta}} q^{NS}(y)dy.$$

where

$$X^{NS}(z(\theta \mid \eta)) \equiv z(\theta \mid \eta)q^{NS}(z(\theta \mid \eta)) + \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(z)dz.$$

The construction of $z(\theta \mid \eta)$ implies that $z(\bar{\theta}(\eta) \mid \eta) \leq \bar{\theta}$ for any $\eta \in \Pi$. Hence

$$X^{NS}(z(\theta \mid \eta)) - z(\theta \mid \eta)q^{NS}(z(\theta \mid \eta)) \geq 0$$

for any $(\theta,\eta) \in K$ and

$$E[u_S(\theta,\eta) \mid \eta] = 0$$

from (A) and (B(v)). Then (u_A,u_S,q) is a WCP allocation satisfying interim PCs. Now we show that this allocation generates a higher payoff to P than the optimal allocation in NS. P's resulting expected payoff conditional on η^* (maintaining the expected payoff conditional on $\eta \neq \eta^*$ unchanged) is:

$$E[V(q^{NS}(z(\theta \mid \eta^*))) - z(\theta \mid \eta^*)q^{NS}(z(\theta \mid \eta^*)) - \int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*].$$

With $E[u_S(\theta, \eta^*) \mid \eta^*] = 0$, this is equal to

$$\begin{aligned}
& E[V(q^{NS}(z(\theta \mid \eta^*))) - z(\theta \mid \eta^*)q^{NS}(z(\theta \mid \eta^*)) - \int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
+ \;& \lambda E[(z(\theta \mid \eta^*) - h(\theta \mid \eta^*))q^{NS}(z(\theta \mid \eta^*)) + \int_{z(\theta|\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*] \\
= \;& E[V(q^{NS}(z(\theta \mid \eta^*))) - [(1-\lambda)z(\theta \mid \eta^*) + \lambda h(\theta \mid \eta^*)]q^{NS}(z(\theta \mid \eta^*)) \\
- \;& (1-\lambda)\int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \mid \eta^*] \\
- \;& \lambda \int_{\bar{\theta}(\eta^*)}^{\bar{\theta}} q^{NS}(z)dz
\end{aligned}$$

On the other hand,

$$E[(\theta - h(\theta \mid \eta^*))q^{NS}(\theta) + \int_{\theta}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*] = 0.$$

P's expected payoff conditional on η^* in the optimal allocation in NS is:

$$\begin{aligned}
& E[V(q^{NS}(\theta)) - \theta q^{NS}(\theta) - \int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
= \;& E[V(q^{NS}(\theta)) - \theta q^{NS}(\theta) - \int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
+ \;& \lambda E[(\theta - h(\theta \mid \eta^*))q^{NS}(\theta) + \int_{\theta}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*] \\
= \;& E[V(q^{NS}(\theta)) - [(1-\lambda)\theta + \lambda h(\theta \mid \eta)]q^{NS}(\theta) - (1-\lambda)\int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
- \;& \lambda \int_{\bar{\theta}(\eta^*)}^{\bar{\theta}} q^{NS}(z)dz
\end{aligned}$$

The difference between two payoffs is

$$
\begin{aligned}
& E[V(q^{NS}(z(\theta \mid \eta^*))) - [(1-\lambda)z(\theta \mid \eta^*) + \lambda h(\theta \mid \eta^*)]q^{NS}(z(\theta \mid \eta^*)) \\
- \quad & (1-\lambda)\int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
- \quad & E[V(q^{NS}(\theta)) - [(1-\lambda)\theta + \lambda h(\theta \mid \eta^*)]q^{NS}(\theta) - (1-\lambda)\int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
= \quad & E[\int_{\theta}^{z(\theta|\eta^*)} [V^{'}(q^{NS}(z)) - \{(1-\lambda)z + \lambda h(\theta \mid \eta^*)\}]q^{NS'}(z)dz \mid \eta^*] \\
= \quad & E[\int_{\theta}^{z(\theta|\eta^*)} [H(z) - \{(1-\lambda)z + \lambda h(\theta \mid \eta^*)\}]q^{NS'}(z)dz \mid \eta^*].
\end{aligned}
$$

The second equality uses $V^{'}(q^{NS}(z)) = H(z)$. From the construction of $z(\theta \mid \eta^*)$ in Step 4 and $q^{NS'}(z) < 0$, this is positive. We have thus found an implementable allocation generating a higher payoff to P in CS compared to the optimal allocation in NS. ■

Proof of Proposition 5:

Since $f(\theta \mid \eta^*)$ is decreasing in θ, $h(\theta \mid \eta^*)$ is increasing in θ, implying $h(\theta \mid \eta^*) = \hat{h}(\theta \mid \eta^*)$. Since $\frac{f(\theta|\eta^*)}{f(\theta|\eta)}$ is strictly decreasing in θ for any $\eta \neq \eta^*$, $\frac{f(\theta^{'}|\eta^*)}{f(\theta|\eta^*)} > \frac{f(\theta^{'}|\eta)}{f(\theta|\eta)}$ for $\theta > \theta^{'}$. $\Theta(\eta) = \Theta(\eta^*) = \Theta$ then implies

$$\frac{F(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} = \int_{\underline{\theta}}^{\theta} \frac{f(\theta^{'} \mid \eta^*)}{f(\theta \mid \eta^*)} d\theta^{'} > \int_{\underline{\theta}}^{\theta} \frac{f(\theta^{'} \mid \eta)}{f(\theta \mid \eta)} d\theta^{'} = \frac{F(\theta \mid \eta)}{f(\theta \mid \eta)}.$$

Hence $h(\theta \mid \eta^*) > h(\theta \mid \eta)$ for $\theta \in (\underline{\theta}, \bar{\theta}]$ and $h(\underline{\theta} \mid \eta^*) = h(\underline{\theta} \mid \eta) = \underline{\theta}$. The ironing procedure then ensures that $\hat{h}(\theta \mid \eta^*) > \hat{h}(\theta \mid \eta)$ for any $\theta > \underline{\theta}$ and any $\eta \neq \eta^*$. Thus $\hat{h}(\bar{\theta}|\eta^*) > \hat{h}(\bar{\theta}|\eta)$ while $\hat{h}(\underline{\theta}|\eta^*) = \hat{h}(\underline{\theta}|\eta) = \underline{\theta}$ for $\eta \neq \eta^*$, i.e., the range of $\hat{h}$ conditional on η^* includes the range of $\hat{h}$ conditional on η. Since $h(\theta \mid \eta^*) = \hat{h}(\theta \mid \eta^*)$ is strictly increasing and continuously differentiable, $q^*(\hat{h}(\theta \mid \eta^*))$ is also continuously differentiable and strictly decreasing in θ.

Suppose the result is false, and the second best allocation

$$(u_A^{SB}(\theta, \eta), u_S^{SB}(\theta, \eta), q^{SB}(\theta, \eta))$$

is implementable with weak collusion. Then Proposition 1 implies existence of $\pi(\cdot \mid \eta) \in Y(\eta)$ such that for any $(\theta, \eta), (\theta', \eta')$,

$$q^{SB}(\theta, \eta) = q^*(\hat{h}(\theta \mid \eta))$$

$$X^{SB}(\theta, \eta) - z(\theta \mid \eta) q^{SB}(\theta, \eta) \geq 0$$

$$X^{SB}(\theta, \eta) - z(\theta \mid \eta) q^{SB}(\theta, \eta) \geq X^{SB}(\theta', \eta') - z(\theta \mid \eta) q^{SB}(\theta', \eta')$$

where $z(\theta \mid \eta) \equiv z(\theta, \pi(\theta \mid \eta), \eta)$ and

$$X^{SB}(\theta, \eta) \equiv u_A^{SB}(\theta, \eta) + u_S^{SB}(\theta, \eta) + \theta q^{SB}(\theta, \eta).$$

Step 1: $z(\theta \mid \eta) \in [z(\underline{\theta} \mid \eta^*), z(\bar{\theta} \mid \eta^*)]$ holds for any (θ, η).

The proof is as follows. Since $\hat{h}(\theta \mid \eta) < \hat{h}(\theta \mid \eta^*)$ for any $\theta > \underline{\theta}$ and $\eta \neq \eta^*$,

$$q^{SB}(\theta, \eta^*) = q^*(\hat{h}(\theta \mid \eta^*)) < q^*(\hat{h}(\theta \mid \eta)) = q^{SB}(\theta, \eta).$$

Then $z(\theta \mid \eta^*) \geq z(\theta \mid \eta)$ follows from the coalitional incentive constraints.

If on the other hand $z(\underline{\theta}|\eta) < z(\underline{\theta}|\eta^*)$, there exists a non-degenerate interval T of θ for which $z(\theta|\eta) \in (z(\underline{\theta}|\eta), z(\underline{\theta}|\eta^*))$. The second-best output in either state $(\underline{\theta}, \eta)$ or $(\underline{\theta}, \eta^*)$ is the first-best level $q^*(\underline{\theta})$ corresponding to cost $\underline{\theta}$. The coalitional incentive constraints imply output must be constant over T given η, so must equal the first-best $q^*(\underline{\theta})$ corresponding to cost $\underline{\theta}$. But $\hat{h}(\theta, \eta) \geq \theta$ for every $\theta \in T$, implying $q^{SB}(\theta, \eta) = q^*(\hat{h}(\theta, \eta)) \leq q^*(\theta) < q^*(\underline{\theta})$, and we obtain a contradiction.

In what follows, we denote $[z(\underline{\theta} \mid \eta^*), z(\bar{\theta} \mid \eta^*)]$ by $[\underline{z}, \bar{z}]$.

Step 2:

Now we claim that there exists $\phi(\cdot) : [\underline{h}, \bar{h}] \to [\underline{z}, \bar{z}]$ which satisfies

(i) $z(\theta \mid \eta) = \phi(\hat{h}(\theta \mid \eta))$.

(ii) $\phi(h)$ is continuous, and non-decreasing in h.

(iii) $h - \phi(h)$ is non-negative and increasing in h.

First we show that for any (θ, η) and (θ', η') such that $\hat{h}(\theta \mid \eta) = \hat{h}(\theta' \mid \eta')$, $z(\theta \mid \eta) = z(\theta' \mid \eta')$. Otherwise, there exists (θ', η') and (θ'', η'') such that $\hat{h}(\theta' \mid \eta') = \hat{h}(\theta'' \mid \eta'')$ and $z(\theta' \mid \eta') \neq z(\theta'' \mid \eta'')$. Suppose $z(\theta' \mid \eta') < z(\theta'' \mid \eta'')$ without loss of generality. By Step 1 and the continuity of $z(\theta \mid \eta^*)$, there exists θ_1 and θ_2 $(\theta_1 < \theta_2)$ such that

$$z(\theta_1 \mid \eta^*) = z(\theta' \mid \eta') < z(\theta'' \mid \eta'') = z(\theta_2 \mid \eta^*).$$

Since $z(\theta \mid \eta^*)$ is continuous in θ and non-decreasing in θ,

$$z(\theta' \mid \eta') \leq z(\theta \mid \eta^*) \leq z(\theta'' \mid \eta'')$$

for any $\theta \in [\theta_1, \theta_2]$. The coalitional incentive constraints imply

$$q^{SB}(\theta', \eta') \geq q^{SB}(\theta, \eta^*) \geq q^{SB}(\theta'', \eta'')$$

for any $\theta \in [\theta_1, \theta_2]$. On the other hand $\hat{h}(\theta' \mid \eta') = \hat{h}(\theta'' \mid \eta'')$ implies $q^{SB}(\theta', \eta') = q^{SB}(\theta'', \eta'')$. Therefore $q^{SB}(\theta, \eta^*) = q^{SB}(\theta', \eta') = q^{SB}(\theta'', \eta'')$ for any $\theta \in [\theta_1, \theta_2]$. This contradicts the property that $q^{SB}(\theta, \eta^*)$ must be strictly decreasing in θ.

Hence there exists a function $\phi(\cdot) : [\underline{h}, \bar{h}] \to [\underline{z}, \bar{z}]$ such that $z(\theta \mid \eta) = \phi(\hat{h}(\theta \mid \eta))$. Since $z(\theta \mid \eta^*)$ and $\hat{h}(\theta \mid \eta^*)$ are continuous in θ, $\phi(h)$ must be continuous.

Second we show that $\phi(h)$ is non-decreasing in h. For any (θ, η) and (θ', η') such that $\hat{h}(\theta \mid \eta) < \hat{h}(\theta' \mid \eta')$,

$$q^{SB}(\theta, \eta) = q^*(\hat{h}(\theta \mid \eta)) > q^*(\hat{h}(\theta' \mid \eta')) = q^{SB}(\theta', \eta').$$

The coalitional incentive constraints then imply $z(\theta \mid \eta) \leq z(\theta' \mid \eta')$.

Third we show $h - \phi(h)$ is non-negative and increasing in h. Since $q^{SB}(\theta, \eta^*) = q^*(\hat{h}(\theta \mid \eta^*))$ is strictly decreasing in θ, the pooling region $\Theta(\pi(\cdot \mid \eta^*), \eta^*)$ must be empty. Hence it must be the case that

$$z(\theta \mid \eta^*) = \theta + \frac{F(\theta \mid \eta^*) - \Lambda(\theta \mid \eta^*)}{f(\theta \mid \eta^*)},$$

implying

$$\frac{\Lambda(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} = \hat{h}(\theta \mid \eta^*) - \phi(\hat{h}(\theta \mid \eta^*)).$$

The LHS is non-negative and increasing in θ, since $f(\theta \mid \eta^*)$ is decreasing in θ and $\Lambda(\theta \mid \eta^*)$ is non-negative and non-decreasing in θ. So $h - \phi(h)$ must be non-negative and increasing in $h \in [\underline{h}, \bar{h}]$.

Step 3:

Define $R(z) \equiv \max_{(\tilde{\theta},\tilde{\eta})\in K}[X^{SB}(\tilde{\theta},\tilde{\eta}) - zq^{SB}(\tilde{\theta},\tilde{\eta})]$ for any $z \in [\underline{z}, \bar{z}]$. Then

$$R(z(\theta \mid \eta)) = X^{SB}(\theta,\eta) - z(\theta \mid \eta)q^{SB}(\theta,\eta)$$

and by the Envelope Theorem, $R^{'}(z(\theta \mid \eta)) = -q^{SB}(\theta,\eta) = -q^*(\hat{h}(\theta \mid \eta))$. It also implies $R^{'}(\phi(h)) = -q^*(h)$. Then S's interim payoff is

$$\begin{aligned}
& E[X^{SB}(\theta,\eta) - h(\theta \mid \eta)q^{SB}(\theta,\eta) \mid \eta] \\
= \ & E[X^{SB}(\theta,\eta) - z(\theta \mid \eta)q^{SB}(\theta,\eta) + (z(\theta \mid \eta) - h(\theta \mid \eta))q^{SB}(\theta,\eta) \mid \eta] \\
= \ & E[R(\phi(\hat{h}(\theta \mid \eta))) + (\phi(\hat{h}(\theta \mid \eta)) - \hat{h}(\theta \mid \eta))q^*(\hat{h}(\theta \mid \eta)) \mid \eta]
\end{aligned}$$

with the last equality using the property of the ironing rule.

Next define

$$L(h) \equiv R(\phi(h)) + (\phi(h) - h)q^*(h).$$

$L(h)$ is continuous and differentiable almost everywhere, since the monotonicity implies the differentiability of $\phi(h)$ almost everywhere. If the second best allocation is implementable with weak collusion, $E[L(\hat{h}(\theta \mid \eta)) \mid \eta] = 0$ holds for any η. The first derivative of $L(h)$ is

$$L^{'}(h) = (\phi(h) - h)q^{*'}(h) - q^*(h).$$

Since $q^*(h)$ is continuously differentiable, $L^{'}(h)$ is continuous and also differentiable almost everywhere and

$$L^{''}(h) = (\phi^{'}(h) - 1)q^{*'}(h) + (\phi(h) - h)q^{*''}(h) - q^{*'}(h).$$

By using $V'(q^*(h)) = h$, we can show that $V'''(q) \leq 0$ implies $q^{*''}(h) \leq 0$, and $0 < V'''(q) \leq \frac{(V''(q))^2}{V'(q)}$ implies $q^{*''}(h) > 0$ and $hq^{*''}(h) + q^{*'}(h) < 0$. By $\phi'(h) - 1 < 0$ and $\phi(h) - h \leq 0$, it follows that $L''(h) > 0$.

The strict convexity of L then implies $L(h) > L(h') - (h' - h)L'(h')$ for any $h \neq h'$. Hence

$$\begin{aligned} & E[L(\hat{h}(\theta \mid \eta^*)) \mid \eta^*] = E[L(h(\theta \mid \eta^*)) \mid \eta^*] \\ > \; & E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - h(\theta \mid \eta^*)]L'(\hat{h}(\theta \mid \eta)) \mid \eta^*] \\ = \; & E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - \theta]L'(\hat{h}(\theta \mid \eta)) + \int_\theta^{\bar{\theta}} L'(\hat{h}(y \mid \eta))dy \mid \eta^*] \end{aligned}$$

for any $\eta \neq \eta^*$. $L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - \theta]L'(\hat{h}(\theta \mid \eta)) + \int_\theta^{\bar{\theta}} L'(\hat{h}(y \mid \eta))dy$ is non-increasing in θ, since

$$-[\hat{h}(\theta \mid \eta) - \theta]L''(\hat{h}(\theta \mid \eta)) < 0$$

and is strictly decreasing in θ over some interval (since the ironing rule ensures $\hat{h}(\theta \mid \eta)$ is continuous with $\hat{h}(\underline{\theta} \mid \eta) = \underline{\theta}$ and $\hat{h}(\bar{\theta} \mid \eta) > \bar{\theta}$). Then property (ii) implies $F(\theta \mid \eta^*) > F(\theta \mid \eta)$ for $\theta \in (\underline{\theta}, \bar{\theta})$ and for any $\eta \neq \eta^*$. A first order stochastic dominance argument then ensures

$$\begin{aligned} & E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - \theta]L'(\hat{h}(\theta \mid \eta)) + \int_\theta^{\bar{\theta}} L'(\hat{h}(y \mid \eta))dy \mid \eta^*] \\ > \; & E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - \theta]L'(\hat{h}(\theta \mid \eta)) + \int_\theta^{\bar{\theta}} L'(\hat{h}(y \mid \eta))dy \mid \eta] \\ = \; & E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - h(\theta \mid \eta)]L'(\hat{h}(\theta \mid \eta)) \mid \eta] \\ = \; & E[L(\hat{h}(\theta \mid \eta)) \mid \eta]. \end{aligned}$$

where the last equality utilizes a property of the ironing transformation. Therefore S must earn a positive rent in state η^*, as $E[L(h(\theta \mid \eta^*)) \mid \eta^*] > E[L(\hat{h}(\theta \mid \eta)) \mid \eta] \geq 0$. This is a contradiction. ■

Proof of Propositions 6, 7, 8: sketched in the text.

Proof of Proposition 9: We start by defining $\bar{H}(\theta)$ on $\theta \leq 1$ such that $\bar{H}(\theta) = 0$ for $\theta \leq 0$ and $\bar{H}(\theta) = H(\theta)$ for $\theta \in (0, 1]$. We also define $\bar{l}_i(\theta)$ and $\bar{h}_i(\theta)$ $(i = 1, 2)$ as $\bar{l}_i(\theta) = l_i(\theta)$ for $\theta \in (\underline{\theta}_i, \bar{\theta}_i]$ and $\bar{l}_i(\theta) = -\infty$ for $\theta = \underline{\theta}_i$, and $\bar{h}_i(\theta) = h_i(\theta)$ for $\theta \in [\underline{\theta}_i, \bar{\theta}_i)$ and $\bar{h}_i(\theta) = +\infty$ for $\theta = \bar{\theta}_i$. Then conditions (8, 9, 10) reduce to

$$\max\{\bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\} \leq b \leq \min\{\bar{h}_1(\theta_1), \bar{h}_2(\theta_2)\}.$$

As a first step, let us show that it is beneficial to hire S for any $V \in (\max\{c, \bar{H}(l_2(c))\}, H(1))$. For any $V \in (\max\{c, \bar{H}(l_2(c))\}, H(1))$, since $h_1(0) = 0 < V$ and $h_2(c) = c < V$, $\theta_1^{SB} > 0$, $\theta_2^{SB} > c$ and $\theta^{NS} \in (0, 1)$ where $V = H(\theta^{NS})$. P's payoff without S is $\Pi_{NS} \equiv F(\theta^{NS})[V - \theta^{NS}] > 0$. In the third-best problem with S, Π_{NS} can be achieved if we select $X_0 = 0$, $b = \theta_1 = \theta^{NS}$ and $\theta_2 = c$ in the case of $c < V < H(c)$, and $X_0 = 0$, $\theta_2 = b = \theta^{NS}$ and $\theta_1 = c$ in the case of $H(c) \leq V \leq H(1)$.

First consider the case $\max\{c, \bar{H}(l_2(c))\} < V < H(c)$. Then we observe the following relationship among the thresholds in NS and SB:

$$\theta_1^{SB} = \theta^{NS} < c < \theta_2^{SB}.$$

Let us create a small variation from the optimal NS ($X_0 = 0$, $b = \theta_1 = \theta^{NS}$ and $\theta_2 = c$) to $(\theta_1', \theta_2', b', X_0')$ which satisfies

(i) $\theta_1' = \theta_1 = \theta^{NS}$

(ii) θ_2' is selected such that $\theta_2' \in (\theta_2, \theta_2^{SB})$ and $F(\theta_1 \mid \eta_1) > F(\theta_2' \mid \eta_2)$

(iii) $b' = \frac{\theta_1 F(\theta_1|\eta_1) - \theta_2' F(\theta_2'|\eta_2)}{F(\theta_1|\eta_1) - F(\theta_2'|\eta_2)} < \theta_1 < \theta_2'$

(iv) $X_0' = F(\theta_1 \mid \eta_1)(\theta_1 - b') = F(\theta_2' \mid \eta_2)(\theta_2' - b') > 0$.

With this variation, the threshold pair moves closer to the second best one, while maintaining S's zero information rent owing to (iv). Hence P's payoff is strictly improved. In order for this allocation to be implementable under weak collusion, we need to check that

$$\max\{l_1(\theta_1), l_2(\theta_2')\} \leq b' \leq \min\{h_1(\theta_1), h_2(\theta_2')\}.$$

It is evident that

$$b' < \theta_1 \leq \min\{h_1(\theta_1), h_2(\theta_2')\}.$$

$\max\{c, \bar{H}(l_2(c))\} < V = H(\theta^{NS})$ implies $l_2(c) < \theta_1 = \theta^{NS}$. Since $\lim_{\theta_2' \to c} b' = \theta_1 = \theta^{NS}$, we can find θ_2' sufficiently close to c such that

$$\max\{l_1(\theta_1), l_2(\theta_2')\} \leq b'.$$

Next consider the case $H(c) \leq V < H(1)$. Then

$$\theta_1^{SB} = c < \theta^{NS} < \theta_2^{SB}$$

Construct the following small variation from the optimal NS allocation $((\theta_1, \theta_2, b, X_0) = (c, \theta^{NS}, \theta^{NS}, 0))$ to $(\theta_1', \theta_2', b', X_0')$ which satisfies

(i) $X_0' = F(\theta_2' \mid \eta_2)(\theta_2' - b')$

(ii) $\theta_1' = \theta_1 = c$

(iii) θ_2' satisfies $\theta^{NS} < \theta_2' < \theta_2^{SB}$

(iv) $b' = \frac{\theta^{NS} - F(\theta_2'|\eta_2)\theta_2'}{(1-F(\theta_2'|\eta_2))} < \theta^{NS}$.

Since $b' < \theta^{NS} < \theta_2'$, $X_0' > 0$, and the coalitional participation constraint is satisfied. $b' \leq \min\{\bar{h}_1(c), \bar{h}_2(\theta_2')\}$ is obviously satisfied. $b' \geq \max\{\bar{l}_1(c), \bar{l}_2(\theta_2')\} = \max\{c, l_2(\theta_2')\}$ is also satisfied for θ_2' sufficiently close to θ^{NS}, since $\lim_{\theta_2' \to \theta^{NS}} b' = \theta^{NS}$ and $\theta^{NS} > \max\{c, l_2(\theta^{NS})\}$. P's payoff is strictly improved with this allocation, since it moves closer to the second best, while S's interim payoff is unchanged as $(b' - c) + X_0' = (\theta^{NS} - c)$ in state η_1 and S earns zero rent in state η_2 owing to (i) above.

To proceed with the necessity part of the result, we establish the following lemmata which help characterize the optimal allocation.

Lemma 4 *In the third-best solution, either $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ or $(\theta_1, \theta_2) = (c, 1)$ holds.*

Proof of Lemma 4

Suppose otherwise that the solution satisfies $F(\theta_1 \mid \eta_1) \leq F(\theta_2 \mid \eta_2)$ and $(\theta_1, \theta_2) \neq (c, 1)$. This implies that $\theta_1 < c$. The objective function of P

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - \max\{0, F(\theta_1 \mid \eta_1)(\theta_1 - b), F(\theta_2 \mid \eta_2)(\theta_2 - b)\}$$

is non-decreasing in b for $b \leq \theta_2$, and is non-increasing in b for $b > \theta_2$, implying that it is maximized at $b = \theta_2$. Feasibility requires that $\max\{\bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\} \leq \theta_2$. Hence $b = \min\{\theta_2, \bar{h}_1(\theta_1), \bar{h}_2(\theta_2)\}$ in the optimal solution, implying P's payoff is:

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b).$$

But this is less than the P's payoff in the optimal solution to NS, since

$$\begin{aligned} & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b) \\ \leq \; & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_2). \\ < \; & F(\theta^{NS})(V - \theta^{NS}) \end{aligned}$$

The first inequality comes from $b \leq \theta_2$ and $F(\theta_1 \mid \eta_1) \leq F(\theta_2 \mid \eta_2)$. The second inequality comes from the fact that (i) if $V > \theta_2$,

$$\begin{aligned} & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_2) < [p_1 + p_2 F(\theta_2 \mid \eta_2)](V - \theta_2) \\ = \; & F(\theta_2)(V - \theta_2) \leq F(\theta^{NS})(V - \theta^{NS}) \end{aligned}$$

and (ii) if $V \leq \theta_2$,

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_2) \leq 0 < F(\theta^{NS})(V - \theta^{NS}).$$

Hence we obtain a contradiction, establishing the Lemma. ■

For (θ_1, θ_2) which satisfies $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$, let us define $B(\theta_1, \theta_2)$ as

$$B(\theta_1, \theta_2) \equiv \frac{\theta_1 F(\theta_1 \mid \eta_1) - \theta_2 F(\theta_2 \mid \eta_2)}{F(\theta_1 \mid \eta_1) - F(\theta_2 \mid \eta_2)},$$

which is the value of b which satisfies $F(\theta_1 \mid \eta_1)(\theta_1 - b) = F(\theta_2 \mid \eta_2)(\theta_2 - b)$. It is evident that $B(\theta_1, \theta_2) \leq \theta_1 \leq \theta_2$. Now let us consider the following problem (P_1):

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b)$$

subject to

$$b = \max\{B(\theta_1, \theta_2), l_1(\theta_1), \bar{l}_2(\theta_2)\}$$

and

$$F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2).$$

Then we can characterize the optimal allocation as follows.

Lemma 5 *If the problem P_1 has a solution (θ_1^*, θ_2^*), it is a pair of thresholds in the optimal allocation associated with*

$$b^* = \max\{B(\theta_1^*, \theta_2^*), l_1(\theta_1^*), \bar{l}_2(\theta_2^*)\}$$

and

$$X_0^* = F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*).$$

If the problem P_1 does not have a solution, $(\theta_1, \theta_2, b, X_0) = (c, 1, 1, 0)$ is the optimal allocation.

Proof of Lemma 5

Suppose that the optimal threshold (θ_1^*, θ_2^*), which is a solution of the third-best problem, satisfies $F(\theta_1^* \mid \eta_1) > F(\theta_2^* \mid \eta_2)$. Then (θ_1^*, θ_2^*) must be also a solution of the revised third-best problem where the constraint $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ is added; we hereafter refer to this as problem $P_0^{'}$. Consider now the solution of $P_0^{'}$. For (θ_1, θ_2) which satisfies the additional constraint $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$, the objective function in $P_0^{'}$:

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - \max\{0, F(\theta_1 \mid \eta_1)(\theta_1 - b), F(\theta_2 \mid \eta_2)(\theta_2 - b)\}$$

is non-decreasing in b for $b < B(\theta_1, \theta_2)$ and is non-increasing in b for $b \in [B(\theta_1, \theta_2), \theta_2]$ and is non-increasing in b for $b > \theta_2$. Therefore it is maximized at $b = B(\theta_1, \theta_2)$. Since $\min\{\bar{h}_1(\theta_1), \bar{h}_2(\theta_2)\} > \theta_1 \geq B(\theta_1, \theta_2)$ for any (θ_1, θ_2) such that $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$, the optimal b must satisfy

$$b = \max\{B(\theta_1, \theta_2), \bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\}.$$

Since $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ also implies $\theta_1 > 0$, we can replace $\bar{l}_1(\theta_1)$ by $l_1(\theta_1)$ without loss of generality. Since $\max\{\bar{l}(\theta_1), \bar{l}_2(\theta_2)\} < \theta_2$ and $B(\theta_1, \theta_2) \leq \theta_2$, this optimal choice of b must be in $[B(\theta_1, \theta_2), \theta_2]$, implying that the optimal choice of X_0 satisfies

$$X_0 = F(\theta_2 \mid \eta_2)(\theta_2 - b).$$

Let L denote the set of (θ_1, θ_2) pairs satisfying the property that $l_2(\theta_2) \leq h_1(\theta_1)$, provided $\theta_1 < c < \theta_2$. Any feasible allocation must satisfy the property that $(\theta_1, \theta_2) \in L$, since conditions (8, 9, 10) require the existence of b such that $b \leq h_1(\theta_1)$ and $b \geq l_2(\theta_2)$ if $\theta_1 < c < \theta_2$. Since $l_1(\theta) = h_2(\theta) = \theta + \frac{F(\theta) - F(c)}{f(\theta)}$ for all θ, it follows that $l_1(\theta_1) \leq h_2(\theta_2)$ is automatically satisfied. Hence the condition $(\theta_1, \theta_2) \in L$ ensures that there exists b such that (b, θ_1, θ_2) satisfies conditions (8, 9, 10).

Hence problem P_0' reduces to

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b)$$

subject to

$$b = \max\{B(\theta_1, \theta_2), l_1(\theta_1), \bar{l}_2(\theta_2)\}$$

$$F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$$

and

$$(\theta_1, \theta_2) \in L.$$

where, by hypothesis, the optimal pair of thresholds (θ_1^*, θ_2^*), is also the solution of this problem.

Next we show that (θ_1^*, θ_2^*) is also the solution of the problem P_1 where the last constraint $(\theta_1, \theta_2) \in L$ is dropped from the above problem. To show it by contradiction, suppose that (θ_1^*, θ_2^*) is the solution of P_1 with additional constraint $(\theta_1, \theta_2) \in L$, but is not the solution of P_1. Then we can find $(\theta_1^{'}, \theta_2^{'}) \notin L$ such that the objective function of P_1 can take a higher value, satisfying $F(\theta_1^{'} \mid \eta_1) > F(\theta_2^{'} \mid \eta_2)$. $(\theta_1^{'}, \theta_2^{'}) \notin L$ implies that $h_1(\theta_1^{'}) < l_2(\theta_2^{'})$, $\theta_2^{'} > c$ and $\theta_1^{'} < c$. Then since

$$\max\{B(\theta_1^{'}, \theta_2^{'}), l_1(\theta_1^{'})\} < \theta_1^{'} \leq h_1(\theta_1^{'}) < l_2(\theta_2^{'}),$$

the choice of b must be $b^{'} = l_2(\theta_2^{'}) \leq \theta_2^{'}$. Since the value of the objective function

$$[p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)](V - l_2(\theta_2^{'})) - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - l_2(\theta_2^{'}))$$

is positive (as P earns a positive payoff under (θ_1^*, θ_2^*)), $V > l_2(\theta_2^{'})$ must be satisfied. Now define $\theta_1^{''} \equiv \max\{\theta_1 \mid h_1(\theta_1) \leq l_2(\theta_2^{'}), \theta_1 \leq c\}$, which is strictly larger than $\theta_1^{'}$. Then $(\theta_1^{''}, \theta_2^{'}) \in L$ and $F(\theta_1^{''} \mid \eta_1) > F(\theta_2^{'} \mid \eta_2)$. Since

$$\max\{B(\theta_1^{''}, \theta_2^{'}), l_1(\theta_1^{''})\} \leq \theta_1^{''} < h_1(\theta_1^{''}) \leq l_2(\theta_2^{'}),$$

the choice of b is still equal to $l_2(\theta_2^{'})$. It is evident that this choice $(\theta_1^{''}, \theta_2^{'})$ generates a higher value of the objective function:

$$\begin{aligned} & [p_1 F(\theta_1^{''} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)](V - l_2(\theta_2^{'})) - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - l_2(\theta_2^{'})) \\ > \quad & [p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)](V - l_2(\theta_2^{'})) - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - l_2(\theta_2^{'})). \end{aligned}$$

Since the left side hand cannot be larger than the maximum value in the problem P_1 with the additional constraint $(\theta_1, \theta_2) \in L$, we obtain a contradiction. We conclude that if (θ_1^*, θ_2^*) which is the solution of the problem P_0 satisfies $F(\theta_1^* \mid \eta_1) > F(\theta_2^* \mid \eta_2)$, the problem P_1 has a solution (θ_1^*, θ_2^*). Hence if P_1 does not have a solution, the solution of P_0 does not satisfy $F(\theta_1^* \mid \eta_1) > F(\theta_2^* \mid \eta_2)$. Then from Lemma 4, $(\theta_1^*, \theta_2^*) = (c, 1)$ is the optimal in P_0.

Finally we show that if P_1 has a solution, it must be always a solution of P_0. Suppose that P_1 has a solution $(\theta_1^{'}, \theta_2^{'})$, but it is not a solution of P_0. Then from

Lemma 4, $(\theta_1, \theta_2) = (c, 1)$ must be solution of P_0 and the P's payoff is $V - 1$. Then with (θ_1', θ_2'), the objective function in the problem P_1 must take strictly lower value than $V - 1$. However the value of the objective function in the problem P_1 can approximate $V - 1$ by selecting (θ_1, θ_2) which is sufficiently close to $(c, 1)$ without violating all the constraints. This is a contradiction. ■

Using these lemmata, we show that if $c < \bar{H}(l_2(c))$, there exists V_1 such that $c < V_1 \leq \bar{H}(l_2(c))$ and S is not valuable for $V \in (c, V_1]$ and valuable for $V \in (V_1, \bar{H}(l_2(c))$. In order to show it by contradiction, suppose that the supervisor is valuable for any $V \in (c, \bar{H}(l_2(c))]$. Then we can show that $c < \theta_2 < 1$ must hold in the optimal allocation for any $V \in (c, H(l_2(c))]$. The argument is as follows. Consider the problem with the restriction to $\theta_2 = c$. Then $B(\theta_1, c) = \theta_1$ and $b = \theta_1$, and the maximum value in the problem P_1 is Π_{NS} under $\theta_1 = \theta^{NS}$. It implies that if S is valuable, we must have $\theta_2 > c$. With the choice of $\theta_2 = 1$, P's possible maximum payoff is $V - 1$ with the choice of $\theta_1 = c$, which is lower than Π^{NS} since $\bar{H}(l_2(c)) < H(1)$, implying $\theta_2 < 1$.

Therefore the following problem (with the additional constraint $\theta_2 > c$) has a solution and its maximum value is larger than Π_{NS} for any $V \in (c, \bar{H}(l_2(c))]$:

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b)$$

subject to

$$b = \max\{B(\theta_1, \theta_2), l_1(\theta_1), l_2(\theta_2)\}$$

and

$$F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2).$$

$$\theta_2 > c.$$

Let (θ_1^*, θ_2^*) be the solution of the above problem. First we show that $\theta_1^* > l_2(c)$ by contradiction. Suppose $\theta_1^* \leq l_2(c)$. Since it implies $\theta_1^* < c < \theta_2^*$,

$$l_1(\theta_1^*) < \theta_1^* \leq l_2(c) < l_2(\theta_2^*)$$

and

$$B(\theta_1^*, \theta_2^*) < \theta_1^* \leq l_2(c) < l_2(\theta_2^*).$$

It implies $b^* = l_2(\theta_2^*)$. The objective function in the above problem takes a value of

$$[p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V - l_2(\theta_2^*)) - F(\theta_2^* \mid \eta_2)(\theta_2^* - l_2(\theta_2^*)).$$

Since this must be larger than $\Pi_{NS} > 0$ and $\theta_2^* > l_2(\theta_2^*)$, $V > l_2(\theta_2^*)$ must hold. But P's payoff can be improved with the small increase in θ_1 from θ_1^* without violating all constraints of the above problem, which is a contradiction.

With $b^* = \max\{B(\theta_1^*, \theta_2^*), l_1(\theta_1^*), l_2(\theta_2^*)\}$ and $F(\theta_1^* \mid \eta_1) > F(\theta_2^* \mid \eta_2)$,

$$F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*) \geq F(\theta_1^* \mid \eta_1)(\theta_1^* - b^*).$$

It implies that

$$\begin{aligned} & [p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V - b^*) - F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*) \\ \leq \; & p_1 F(\theta_1^* \mid \eta_1)(V - \theta_1^*) + p_2 F(\theta_2^* \mid \eta_2)(V - \theta_2^*). \end{aligned}$$

Then P's payoff in the optimal allocation cannot be larger than the maximum value of the problem:

$$\max p_1 F(\theta_1 \mid \eta_1)(V - \theta) + p_2 F(\theta_2 \mid \eta_2)(V - \theta_2)$$

subject to

$$\theta_1 \geq l_2(c)$$

$$\theta_2 \geq c.$$

Let $\bar{\Pi}(V)$ be the maximum value of the above problem, and $\Pi_{NS}(V)$ be the optimal payoff in NS for V. It is evident that both $\bar{\Pi}(V)$ and $\Pi_{NS}(V)$ are continuous in V. By hypothesis, $\bar{\Pi}(V) > \Pi_{NS}(V)$ for any $V \in (c, \bar{H}(l_2(c))]$. But $\lim_{+V \to c} \bar{\Pi}(V) = F(l_2(c))[V - l_2(c)] < \lim_{+V \to c} \Pi_{NS}(V)$, since $\theta^{NS} < l_2(c)$ at $V = c$. This is the contradiction, implying that there exists some interval of V on $(c, \bar{H}(l_2(c))]$ such that S is not valuable.

Next we show that if there exists $V \in (c, \bar{H}(l_2(c)))$ such that S is valuable, S is also valuable for any $V^{'} \in (V, \bar{H}(l_2(c))]$. Otherwise, suppose there exists $V^{'} \in (V, \bar{H}(l_2(c))]$ such that $(\theta_1, \theta_2) = (\theta^{NS}(V^{'}), c)$ is the solution of P_1, even though (θ_1^*, θ_2^*), which is the solution of P_1 for V satisfies $\theta_1^* > l_2(c)$ and $\theta_2^* > c$ (by the reason explained above). It implies that

$$\begin{aligned} & [p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V - b^*) - F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*) \\ > \ & p_1 F(\theta^{NS}(V^{'}) \mid \eta_1)(V - \theta^{NS}(V^{'})). \end{aligned}$$

and

$$\begin{aligned} & p_1 F(\theta^{NS}(V^{'}) \mid \eta_1)(V^{'} - \theta^{NS}(V^{'})) \\ > \ & [p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V^{'} - b^*) - F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*). \end{aligned}$$

This implies

$$p_1 F(\theta^{NS}(V^{'}) \mid \eta_1) > p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2).$$

But this is inconsistent with $\theta^{NS}(V^{'}) < l_2(c) < \theta_1^*$ and $c < \theta_2^*$, a contradiction. This argument guarantees the existence of a critical value V_1 of V in $(c, \bar{H}(l_2(c))]$, such that S is not valuable (or valuable) for lower (or higher) V than V_1.

Finally let us show that there exists a critical value of V, V_2, with $V_2 \geq H(1)$ such that S is not valuable for V higher than V_2. Otherwise suppose that S is valuable for any $V \geq H(1)$. This implies that P_1 has a solution with $\theta_2^* < 1$ for any $V \geq H(1)$ and its maximum value is higher than $V - 1$. Let $(\theta_1^*, \theta_2^*, b^*)$ be the solution of P_1 for $V \geq H(1)$. Then

$$\begin{aligned} & [p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V - b^*) - F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*) \\ \leq \ & [p_1 + p_2 F(\theta_2^* \mid \eta_2)](V - l_2(\theta_2^*)) - F(\theta_2^* \mid \eta_2)(\theta_2^* - l_2(\theta_2^*)), \end{aligned}$$

since $V \geq H(1) > l_2(\theta_2^*)$, $b^* \geq l_2(\theta_2^*)$ and $F(\theta_1^* \mid \eta_1) > F(\theta_2^* \mid \eta_2)$. By hypothesis, there must exist $\theta_2^* < 1$ such that

$$[p_1 + p_2 F(\theta_2^* \mid \eta_2)](V - l_2(\theta_2^*)) - F(\theta_2^* \mid \eta_2)(\theta_2^* - l_2(\theta_2^*)) > V - 1$$

for any $V \geq H(1)$. It also implies that there exists $\theta_2 < 1$ such that

$$\frac{1 - [p_1 + p_2 F(\theta_2 \mid \eta_2)] l_2(\theta_2) - F(\theta_2 \mid \eta_2)(\theta_2 - l_2(\theta_2))}{p_2[1 - F(\theta_2 \mid \eta_2)]} > V$$

for any $V \geq H(1)$. But this is impossible, since the left hand side is bounded above on $[c, 1]$, because $f(\theta \mid \eta_2)$ is continuous on $[c, 1]$ and

$$\lim_{\theta_2 \to 1} \frac{1 - [p_1 + p_2 F(\theta_2 \mid \eta_2)] l_2(\theta_2) - F(\theta_2 \mid \eta_2)(\theta_2 - l_2(\theta_2))}{p_2[1 - F(\theta_2 \mid \eta_2)]} = 1 + \frac{1}{f(1)}$$

by using l'Hopital's rule. This is a contradiction. Therefore if V is sufficiently large, S cannot generate any value. Finally it is easy to show that if S is not valuable for some $V \geq H(1)$, the same must be true for any larger V, since we can make sure that $p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)$ is monotone for V by the same method as in the previous paragraph. This guarantees the existence of the critical value of $V_2 \geq H(1)$. ■

Proof of Proposition 10: sketched in the text.