Capital and Inequality in the Long Run:

Automation Without Technical Progress[1]

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v1.0, December 2017

Abstract

We present a theory of long run inequality and automation driven by capital accumulation rather than technical progress. Rising capital-labor ratios lower the prices of "robots," which then displace human workers to different degrees in heterogeneous production sectors. Under a singularity condition on the technology of the robot-producing sector, humans eventually get displaced by robots in the production of robots. Thereafter, rising outputs can be produced using capital and robots alone, leading to progressive automation of all other sectors in the economy, and causing capital's share in national income to approach 100%. At the same time, real wages rise in the long run in the absence of barriers to inter-sectoral mobility. However, when the singularity condition does not hold, the robot sector and some final goods sectors are forever protected from automation, allowing (human) labor to retain a positive fraction of income. Our theory is backed by existing empirical evidence, and explains how capital's share of income can rise following capital accumulation despite low capital-labor elasticities of substitution in all sectors.

1. INTRODUCTION

The growing evidence for a declining labor share in income worldwide over the past few decades (Karabarbounis and Neiman 2014, Piketty 2014) has generated active debate concerning possible explanations. Piketty's "$r > g$" theory has been severely criticized (Acemoglu and Robinson 2015, Mankiw 2015, Ray 2015). Explanations in terms of capital-labor substitution along some aggregate CES production function have also been subject to controversy: e.g., standard theories require capital-labor substitution elasticities exceeding one, rejected by panel studies of industry level production functions (Chirinko and Mallick 2014).

An important driver of these distributional changes is the growing trend toward automation in the workplace, commonly attributed to capital-augmenting technical progress. In this paper we seek to draw attention to a different source of automation: the falling relative price of capital prices that arises from ongoing capital accumulation. To highlight this channel, we develop a model which abstracts away altogether from technical progress. Instead, labor replacement

[1]Mookherjee thanks the Department of Economics at NYU for hosting his visit in Fall 2017 when this paper was written. Ray acknowledges funding from the National Science Foundation under grant SES-1629370.

is driven by endogenous changes in relative prices of automated technology (*robots*, in short), which are consequences of the sustained rise in the economy wide capital-labor ratio. Robots can substitute for human labor in the performance of operating tasks, which are combined with capital to produce various intermediate or final goods. We impose no restrictions on substitution elasticity between labor and capital *within* any sector. Sectors differ with respect to their relative proneness to automation, owing to differences in the relative efficiency of human labor *vis-a-vis* robot labor. We abstract from biased technical progress by assuming these differences are exogenously given and unchanging over time.

There is empirical evidence for the channel that we focus on: Karabarbounis and Neiman (2014) show that a substantial fraction of the decline in labor share worldwide is explained by declining capital good prices, *even after controlling for capital-augmenting technical progress, markup rates and skill composition of the labor force.* Each of the latter factors play some role in explaining the distributional changes, but their analysis shows that declining capital goods prices *per se* have also played an independent, significant role.

To explain the logic of this channel: capital accumulation causes the price of capital to fall relative to labor. Automation technologies (robots) are created by capital and human labor, and therefore the prices of such services will be sandwiched between the return to capital and human wages. The question is whether robot prices will be tied more closely to human wages or to the price of capital as capital continues to accumulate. We argue that a crucial "singularity condition" determines the answer to this question. If the robot sector can itself be "robotized" eventually, for which we provide conditions on the underlying technology — the price of robots becomes tied fully to that of capital, causing sustained automation and pushing the share of capital to approach 100% in the long run. This argument applies irrespective of whether within-sector capital-labor substitution elasticities exceed or fall below one, specific patterns of demand, heterogeneity of skills or opportunities for human capital accumulation. In short, the economy-wide substitution elasticity exceeds one, even if it is less than one *within* every sector, owing to the changing composition between automated and non-automated sectors.[2]

In contrast to many existing models of directed technical change via conscious R&D strategies chosen by capitalists, this is a simpler and, in a sense, a more primitive theory. Automation is a consequence of myopic profit-maximizing decisions as capital is progressively accumulated per capita, provided that the singularity condition applies. As in 19th century classical theories, we focus on the effect of ongoing capital accumulation *per se* on the long run distribution of income between capital and labor. Our treatment of labor as a fixed factor runs parallel to Ricardo's theory of rent arising from the status of land as a fixed factor, the relative returns to which rise as other inputs continue to expand around it. Profit maximizing behavior must then favor the choice of technologies that conserve on land needed in industrial production, which in turn limits the growth of rents and frees industrial growth from the stranglehold of land fixity. In our world, it is labor that is the fixed factor, and capital that accumulates. This raises wages, making labor more expensive as an input in production. In the presence of the singularity condition, the resulting upward movement in wages motivates firm owners to automate production.

[2]Alternatively, the model can be applied at the level of a specific industry, which contains firms that vary in their proneness to automation: industry-level substitution elasticities can then exceed one as an increasing proportion of firms automate, even if they are below one within any given firm.

Without the singularity condition, it may still be true that workers in downstream industries face layoffs or stagnating wages. But the rising demand for robots will raise employment and wages in the upstream high-tech sectors. This offset vanishes in the presence of "self-reproducing automata" (von Neumann 1966): robots combine with increasing flows of capital to produce more efficient robots, which in turn produce even more efficient robots and so on, thereby ensuring perennial robot creep into every sector in the economy. The von Neumann singularity is not merely a hypothetical possibility: e.g.,

> "They are a dream of researchers but perhaps a nightmare for highly skilled computer programmers: artificially intelligent machines that can build other artificially intelligent machines.
>
> With recent speeches in both Silicon Valley and China, Jeff Dean, one of Google's leading engineers, spotlighted a Google project called AutoML. ML is short for machine learning, referring to computer algorithms that can learn to perform particular tasks on their own by analyzing data. AutoML, in turn, is a machine-learning algorithm that learns to build other machine-learning algorithms.
>
> With it, Google may soon find a way to create A.I. technology that can partly take the humans out of building the A.I. systems that many believe are the future of the technology industry."
>
> *New York Times, Technology section, November 5, 2017*

The singularity condition that we derive in this paper may be of some intrinsic interest, apart from its implications. It compares two objects: the elasticity of substitution of capital for operating tasks (or labor for short) performed by humans or their robot equivalents, and the efficiency of robots relative to humans in providing labor. The condition is automatically met if the former elasticity equals or exceeds 1, but it is consistent with sub-unitary elasticities of substitution, the exact magnitude depending on the efficiency of robot-human substitution in the robot sector. It should be noted that while the singularity condition has economy-wide ramifications, it is a condition on the robot sector *alone* — we place no restriction on substitution elasticities in any other sector.

In summary, we examine the long run consequences of capital accumulation with a stationary population. Workers can respond to changes in the labor market by investing in skills needed to move to yet-to-be-automated sectors with higher wages. We establish the following results.

First, if the singularity condition holds, in the long run all final good sectors will eventually get automated, though of course at any specific date there will exist sectors that are yet to be automated (Proposition 1).

Next, Proposition 2 describes the distributional consequences of such automation. Initially, there could be upward pressure on human wage in the robot-producing sector, thereby transferring to workers some of the gains from downstream automation. But if the singularity condition holds, this process will come to a halt at some stage, and the benefits from downstream automation then no longer accrue to workers. Via this roundabout process, capital substitutes for labor in the economy as a whole. Effectively, capitalists can appropriate a larger share of incomes generated within the robot sector, and capital's share of national income approaches 100%. The singularity condition is also necessary for these results. If the robot sector never gets automated, labor's share of earnings in that sector does not vanish; hence labor manages to appropriate a positive share of income in the long-run through its growing earnings in the robot sector. This effect is augmented by lack of automation in final good sectors where humans are more difficult to displace compared to the robot sector.

The third result (Proposition 3) describes the implications for wages as capital is progressively accumulated. As each final good becomes automated, the wage in that sector is tied down by the cost of robots in that sector, and there is potential displacement of human workers. The cost of such displacement rises with the relative efficiency of humans *vis-a-vis* robots, and so there is upward pressure on some wages, at least, in order to generate further automation. Indeed, in the absence of restrictions on worker mobility across sectors, there is a single economy-wide wage rate which tends eventually to the upper bound of relative human efficiency across all sectors. With specific assumptions such as Cobb-Douglas preferences and technology, capital accumulation and progressive automation raises the real wage rate monotonically over time (this monotonicity may not obtain in more general settings), at the same time that the share of labor in income is declining.

Our model abstracts from a number of complications for the sake of clarity. (Section 2 describes related literature in more detail.) As already mentioned, we abstract from technical progress so as to throw light on the effects of capital accumulation *per se*. In this respect our approach is distinguished from the models in Acemoglu and Restrepo (2017). We do not model the consumption-saving decisions of households, and the resulting process of capital accumulation. These are inessential to our results. All we need is that in the long run the net savings rate is bounded away from zero, thereby ensuring that per-capita capital grows without bound. If long-run capital is bounded (by rising depreciation rates, dwindling saving rates or limits to environmental resources), our results can be interpreted as approximations to outcomes realized when the bound is sufficiently large. Moreover, we focus on economies with a fixed set of goods, and do not allow for emergence of new sectors. To the extent that new goods constantly appear in which humans are (at least temporarily) more efficient than robots, labor could retain a positive share of income in the long run even if the singularity condition holds.

Finally, as in 19th century classical theories, we focus on the *functional* distribution of income between capital and labor, rather than the distribution of personal incomes (which obviously depends on household investments in different kinds of capital). Whether any given household will be able to earn incomes on par with the rest of the economy will depend on the extent to which they either become capital owners, or move into sectors where humans are more efficient relative to robots. The simplicity and tractability of the model may allow it to be useful in analyzing effects of fiscal policies such as capital taxes, education subsidies, universal basic income or other means to address the distributional consequences of automation.

2. Related Literature and Empirical Evidence

Karabarbounis and Neiman (2014) provide evidence of a significant decline in labor share of income (of the order of 5 percentage points) worldwide since the early 1980s, which occurred within a large majority (6 out of 10) of industries. They show this can be explained by a decline in the relative price of capital goods (of the order of 25% since 1975), attributed to advances in information and communication technology. Using cross-country panel data at the economy-wide as well as one-digit industry level, they estimate a capital-labor substitution elasticity of 1.25. Their results are unaffected by controlling for markup rates, capital-augmenting technical change or skill composition of the labor force. In the model they present to explain their results, however, the decline in the price of capital goods is explained by exogenous capital-augmenting technical progress.

Our model can explain these findings, where the economy (or industry) is interpreted as a composite of different sectors with differential proneness to automation. Our model predicts a decline in the price of capital goods arising from capital accumulation, even in the absence of any technical progress. Consistent with this, Chirinko and Mallick (2014) show a stable negative relationship between changes in the the long run relative price of capital and growth in capital output ratios within US industries between 1975-2000. The substitution elasticity within specific sectors could be less than one, which is what Chirinko and Mallick find at a lower level of aggregation (2-digit industry level).

Models based on exogenous or endogenous technical progress could also account for rising automation, provided they are sufficiently capital-biased. Acemoglu and Restrepo (2017) provide a model of endogenous technical progress where capital-biased shifts in the automation frontier within the range of existing tasks can co-exist with labor-biased creation of new tasks. They show the existence of balanced growth paths for suitable parameter values, where both kinds of technical progress co-exist in the long run. For other parameter values (low enough rates of time preference of households) which generate high rates of capital accumulation, the forces of automation dominate, similar to our model. Our model is considerably simpler than theirs; our approach which focuses on effects of capital accumulation *per se* complementary to theirs which focuses on effects of technical progress. As mentioned above, the value of our approach is that it helps explain the findings of Karabarbounis and Neiman (2014) which are robust to controlling for capital-augmenting technical progress.

Finally, alternative explanations for declining labor share is provided by Autor *et al* (2017) in terms of heterogeneity of firms within industries in markup rates or ratio of fixed non-labor costs to fixed labor costs which affect labor's share of value added. Increasing competition shifts the composition of industries in favor of 'superstar' firms with higher markup rates or fixed non-labor costs, thereby lowering the share of labor at the industry level though not at the firm level. This explanation is complementary to ours. As Karabarbounis and Neiman (2014) show, higher markup rates do contribute to explaining some of the decline in labor's share, but they are orthogonal to the role of declining prices of capital goods that they document.

3. MODEL

3.1. **Technology and Preferences.** There is a continuum of final goods indexed by $i \in (0, 1)$, each produced according to

$$y_i = f_i(k_i, \ell_i) \tag{1}$$

where f_i is an increasing, smooth, strictly quasi-concave, linearly homogeneous production function with $f_i(k, 0) = f_i(0, \ell) = 0$ and exhibiting continuity in the index i. The input k stands for capital or machine input *not* including robots, and ℓ stands for labor, representing operating tasks traditionally performed by humans (h), but which can be automated by the use of robots (r). Each unit of human labor h is equivalent to one unit of labor ℓ, while each unit of robot labor r is worth $\frac{1}{\lambda_i}$ units of labor ℓ in sector i. Hence λ_i denotes the efficiency of human labor relative to robot labor in sector i. This is determined by the level of (exogenous) technical progress. Since we seek to abstract from the presence of such progress, we treat λ_i as a technological parameter, fixed once and for all for any given sector. We then have $\ell_i = (r_i/\lambda_i) + h_i$.

While the assumption of linear substitutability of robots for humans simplifies the analysis, it is not essential. Our results will extend to contexts of less than perfect substitutability where one factor may cease to be used if its relative price is sufficiently high. Note also that humans may continue to work in automated sectors in our model, provided they are willing to work at a wage which renders employers indifferent between them and robots.[3]

Index sectors so that λ_i is nondecreasing in i, so that by construction, sectors with lower i are ones where robots are intrinsically more capable of displacing humans. For ease of exposition, assume that λ_i is continuous and strictly increasing in i. We assume that no sector can be freely automated or is fully protected; that is, $\lambda_i \in (0, \infty)$ for all $i \in [0, 1]$.

Robots are produced in a separate sector r with technology

$$y_r = f_r(k_r, \ell_r), \tag{2}$$

with $\ell_r = (r_r/\lambda_r) + h_r$, where $\lambda_r \in (0, \infty)$ is the efficiency of humans relative to robots.[4] Hence humans could be displaced from the production of robots if robot prices fall enough relative to wages. We call this scenario the *von Neumann singularity*.

There is also an education sector with a corresponding production function

$$y_e = f_e(k_e, \ell_e), \tag{3}$$

with $\ell_e = (r_e/\lambda_e) + h_e$, and $\lambda_e \in (0, \infty)$. As in any other sector, human teachers could be displaced by "robots" (say, MOOCs). This assumption is not essential to our results, and is used only to derive implications for wage dynamics when labor is able to move across sectors.

In both these sectors, as in every final goods sector, the production function is assumed to be increasing, smooth, strictly quasi-concave, and constant returns to scale, with both inputs needed for positive production. The index j will be used as generic notation for a final goods sector as well as the robot or education sector, while we reserve the index i for a final good. The set of final good sectors is denoted by $\mathcal{I}$, and the set of all sectors by $\mathcal{S} \equiv \mathcal{I} \cup \{r, e\}$.

Finally, on the preferences side, all households share the same homothetic utility function

$$U = U(\mathbf{y}) \tag{4}$$

[3] This will happen, for instance, if workers have specific skills which tie them to particular sectors, and are limited in their ability (at least in the short run) to move to other sectors. Then automation will be accompanied by continued employment of those workers, while limiting their subsequent wage growth as the sector continues to expand.

[4] We hope that the slightly odd notation r_r will not cause any confusion.

where $\mathbf{y}$ is the output vector of final goods. By homotheticity, expenditure shares on each final good are independent of income distribution, though they could depend on final goods prices $\mathbf{p} = \{p_i\}$. For our purposes it will suffice to consider only bounded, continuous price functions. An expenditure share function $\{\beta(\mathbf{p})\}$ maps $\mathbf{p}$ to a distribution of nonnegative expenditures over goods — with total income normalized to one — so that $\int_0^1 \beta_i(\mathbf{p})di = 1$ for any bounded continuous $\mathbf{p}$. (Thus the demand for good i by a person with income y is given by $\beta_i(\mathbf{p})y/p_i$.) Assume that $\beta_i(\mathbf{p})$ is positive for all i and continuous in i whenever $\mathbf{p}$ is continuous, positive everywhere and bounded, and for each i $\beta_i(\mathbf{p})$ is continuous in all bounded $\mathbf{p}$ (in the sup norm).

3.2. **Prices and Unit Costs.** We use the capital good as numeraire, so the price of capital will be set at 1. The other prices are $(\mathbf{p}, p_r, p_e)$, which is the price system for all final goods, robots and education, and $(\{\omega_i\}, \omega_r, \omega_e)$, which is the wage system in all sectors (with free mobility of labor, these wages would all be the same, but not otherwise).

By constant returns to scale and the assumption of a price-taking competitive economy, we know that all prices will be pinned down by unit costs of production. That is, output and input prices are connected by:

$$p_j = c_j(1, \nu_j), \tag{5}$$

where 1 is the normalized return to capital, $\nu_j \equiv \min\{\omega_j, \lambda_j p_r\}$ is the effective price of the labor input in sector j, and c_j is the *unit cost function*, dual to the production function f_j. This cost function has standard properties, which we shall have occasion to invoke below.

3.3. **Factor Endowments and Labor Supply.** There is a given endowment $\bar{K}$ of physical capital, which moves freely across sectors. Robot services are produced within the period and are not part of the physical capital endowment. Note that the physical capital or hardware embodied in robots constitute part of the capital endowment of the economy. The production function y_r describes how robot or IT services are produced from the robot or computer hardware, in combination with operating tasks performed either by humans or computers.

The other endowment is human labor. We normalize the total population to 1. We want to accommodate a variety of human capital models. We allow humans to differ in the skills they are born with. Those skills could be sector-specific. So we take as given some initial allocation of the population across sectors, and also invent a place-holding "null sector" where individuals without any particular initial proclivity can be placed. (Our results are completely independent of these initial assignments.) Upon acquiring education, an individual can move from one sector to another, hereafter providing one unit of labor to that sector with no disutility. To move from sector j to j' a person needs $e_{jj'}$ units of education, which is assumed to be continuous in the indices j and j'. Education is purchased from the education sector (already described) at a unit price of p_e.[5] Therefore equilibrium wages and human allocation must satisfy the *necessary* condition that if any sector j is occupied by human labor, then for each alternative sector i, there

[5]If initial skills are not sector-specific, then everyone is in the null sector to start with, and education requirements can be written only as a function of the "destination sector" j'. We can also accommodate "utility costs" of education that are nonlinear in baseline wages, so as to mimic imperfect or missing capital markets.

must exist some sector m such that it is preferable to move from m to j rather than from m to i:

(6) $$\omega_j - p_e e_{mj} \geq \omega_i - p_e e_{mi}.$$

If this (weak) necessary condition is not satisfied, it is not possible for sector j to have a positive supply of human labor. As we shall see, our main results concerning the dynamics of automation and income distribution between labor and capital do not depend on further explicit restrictions on sectoral choice, or how workers are allocated across sectors at the beginning of any date.

Finally, we presume that moving to the null sector is costless. That is, an individual can drop out of the labor market and earn $\underline{\omega} > 0$ on her own. This ensures a positive minimum wage in the labor market in every sector.[6]

3.4. **The Demand for Labor and Capital Inputs.** Given capital endowment $\bar{K}$ and human labor allocation across sectors $h_i, i \in \mathcal{S}$, national income is given by

(7) $$Y = \bar{K} + \int_0^1 \omega_i h_i + \omega_r h_r + \omega_e h_e,$$

recalling again that robots (or education) are competitively produced under constant returns to scale and cannot contribute to national income over and above factor payments to labor and capital. Subtracting off the payments for education, market clearing requires

(8) $$p_i y_i = \beta_i [Y - p_e y_e]$$

for every final good sector i, where each p_i is given by (5).

Now turn to factor demands. In each sector j, labor demand (human or non-human) must satisfy

$$\nu_j = p_j \frac{\partial f_j(k_j, \ell_j)}{\partial \ell_j} = \alpha_j(\nu_j) \frac{p_j y_j}{\ell_j},$$

where $\alpha_j(\nu_j)$ denotes the elasticity of output with respect to the labor input. Unless the production function is Cobb-Douglas, this elasticity will depend on relative factor prices, which are summarized by ν_j. Rearranging,

(9) $$\ell_j = \alpha_j(\nu_j) \frac{p_j y_j}{\nu_j}$$

for every sector $j \in \mathcal{S}$. By exactly the same argument, and remembering that the elasticity of production with respect to capital is $1 - \alpha_j(\nu_j)$, we have for every sector $j \in \mathcal{S}$,

(10) $$p_j \frac{\partial f_j(k_j, \ell_j)}{\partial k_j} = 1,$$

which can be restated as

(11) $$k_j = [1 - \alpha_j(\nu_j)] p_j y_j.$$

[6]The minimum wage $\underline{\omega}$ is denominated in units of capital. Given our assumptions, all prices will be bounded relative to that of the capital good, so it does not really matter just how we denominate this minimum wage.

3.5. **Automation and the Demand for Humans and Robots.** Now we pay more attention to automated and non-automated sectors. Given linear substitution between robots and humans within any production sector, it is immediate that all sectors j with $\omega_j < \lambda_j p_r$ will not be automated. If h_j denotes the demand for humans and r_j the demand for robots/automation in sector j, then in such sectors, (9) translates into a demand for humans, so that

$$h_j = \alpha_j \frac{p_j y_j}{\omega_j} \text{ and } r_j = 0. \tag{12}$$

For a "partially automated sector" j we have $\omega_j = \lambda_j p_r$, and so

$$h_j + \frac{r_j}{\lambda_j} = \alpha_j \frac{p_j y_j}{\omega_j} = \alpha_j \frac{p_j y_j}{\lambda_j p_r}. \tag{13}$$

with firms indifferent across all such combinations of h_j and r_j. Finally, for "fully automated" sectors in which $\nu_j = \lambda_j p_r < \omega_j$, equation (9) translates into a demand for robots:

$$h_j = 0 \text{ and } r_j = \alpha_j \frac{p_j y_j}{p_r}. \tag{14}$$

A sector is *automated* if it uses robots; i.e., it is either partially or fully automated.

3.6. **Equilibrium.** Each final goods market clears by equation (8). In addition, supply has to equal demand for the two intermediate goods (robots and education), as well as capital, robots and human labor within each sector. For the robot sector as a whole, the equilibrium condition is

$$y_r = \int_0^1 r_i di + r_r + r_e, \text{ where } \{r_j\} \text{ satisfies equations (12)–(14).} \tag{15}$$

For capital we recall (11) to write the equilibrium condition as

$$\bar{K} = \int_0^1 [1 - \alpha_i(\nu_i)] p_i y_i di + [1 - \alpha_r(\nu_r)] p_r y_r + [1 - \alpha_e(\nu_e)] p_e y_e. \tag{16}$$

Finally the market for human labor has to clear, with demands given by equations (12)–(14), and supplies by the initial assignment of the population across sectors, along with educational choices that maximize wages net of educational costs for every worker, for which (6) is necessary. Additional conditions are needed to describe the human labor supply correspondence across sectors, but these are cumbersome and are not needed for any of the arguments.

By Walras' Law, we can drop the equilibrium condition for the remaining (education) sector.

3.7. **The von Neumann Singularity and Long-Run Automation.** We state our main result:

PROPOSITION **1.** *Suppose that the following condition holds on the robot sector:*

$$\lambda_r \lim_{\rho \to 0} c_r(\rho, 1) < 1. \tag{17}$$

Then, there exists $K^* > 0$ such that for all $\bar{K} \geq K^*$, the relative price of robots to capital is fully pinned down at some number p_r^* independent of $\bar{K}$, and the robot sector itself becomes automated. Moreover, as $\bar{K} \to \infty$, almost every final goods sector becomes automated as well.

If (17) *fails, then the robot sector remains protected for all* $\bar{K}$*, and in general, so do some final goods sectors. For instance, if there is costless mobility of labor across sectors, then all goods* j *with* $\lambda_j \geq \lambda_r$ *are also protected irrespective of the value of* $\bar{K}$*.*

To understand the proposition, we begin by parsing the singularity condition (17). The cost function $c(\rho, 1)$ is to be interpreted as the unit cost of production when the cost of labor (human or robotic) is set to 1, and the cost of physical capital is ρ. The condition sends ρ to zero. If the production function for the robot industry (defined on labor and capital) has an elasticity of substitution *at least* 1, the implications are clear: the unit cost of production will go to zero with ρ. In that case (17) is trivially satisfied. On the other hand, $c_r(\rho, 1)$ will have a strictly positive limit when the elasticity of substitution is below 1: there cannot be enough substitution away from the expensive input for it not to matter.

As an example, assume f_r is CES with elasticity parameter $\sigma > 0$; that is,

$$y_r = \left[a_k k_r^{\frac{\sigma-1}{\sigma}} + a_\ell \ell_r^{\frac{\sigma-1}{\sigma}} \right]^{\sigma/(\sigma-1)}$$

for parameters $(a_k, a_\ell) \gg 0$ with $a_k + a_\ell = 1$. Then

$$c_r(\rho, \nu) = \left[a_k^\sigma \rho^{1-\sigma} + a_\ell^\sigma \nu^{1-\sigma} \right]^{1/(1-\sigma)},$$

so that

$$\lim_{\rho \to 0} c_r(\rho, 1) = \lim_{\rho \to 0} \left[a_k^\sigma \rho^{1-\sigma} + a_\ell^\sigma \right]^{1/(1-\sigma)}.$$

It is easy to verify that this limit equals zero whenever $\sigma \geq 1$, so that in the CES case, (17) is equivalent to the condition

(18) $$\text{Either } \sigma \geq 1, \text{ or } \sigma \in (0,1) \text{ and } \lambda_r < a_\ell^{\sigma/\sigma-1}.$$

In summary, condition (17) is *always* satisfied if the production function in the robot sector exhibits at least unit elasticity of substitution between capital and labor. It is also consistent with a positive elasticity smaller than one. Indeed, even with a Leontief technology where $\sigma = 0$, the condition requires $\lambda_r < 1/a_\ell$, where a_ℓ is the number of (human-equivalent) labor units needed to produce one robot.

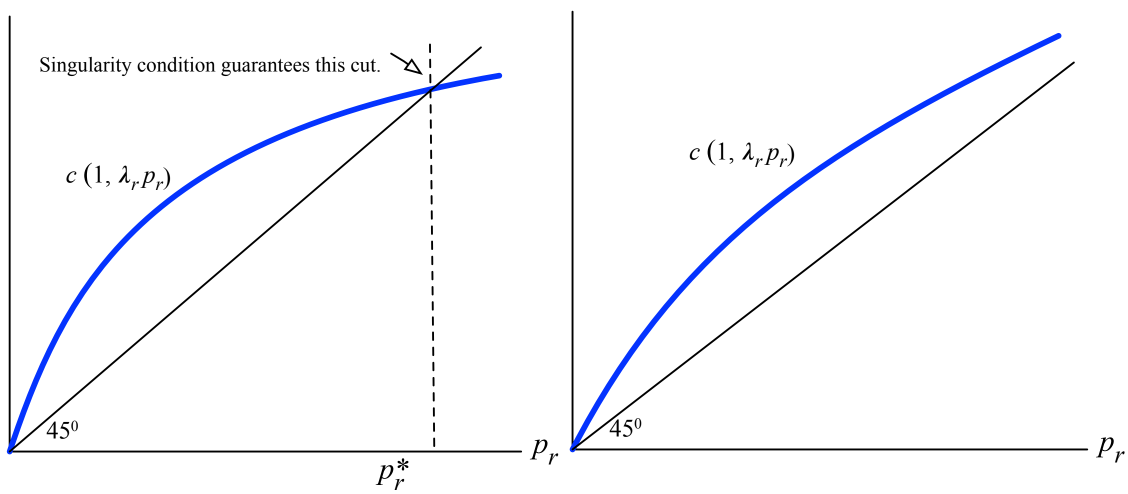
Turning now to the automation implications of the singularity condition (17), notice that in any equilibrium,

(19) $$p_r = c_r(1, \nu_r) \leq c_r(1, \lambda_r p_r),$$

the inequality holding, of course, because $\nu_r \leq \lambda_r p_r$. That inequality *could* hold strictly for every equilibrium, which is just another way of saying that automation in the robot sector cannot occur. However, if the singularity condition does hold, then — provisionally viewing p_r as a parameter — the curve $c_r(1, \lambda_r p_r)$ must ultimately dip below the 45^0 line; see Panel A of Figure 1. Now we see that strict inequality can only hold at all capital stocks if, defining

(20) $$p_r^* = c_r(1, \lambda_r p_r^*),$$

p_r perennially lies *strictly* below p_r^*. But it can't. Imagine a sequence of economies along which the capital stock is exogenously increased without bound. Consumer demand must grow *ad*



(A) Singularity Holds (B) Singularity Fails

FIGURE 1. Condition (17) and the Non-Substitution Bound on Robot Prices

infinitum with capital, because the bound on robot prices just derived also serves to bound all other goods prices via the inequality

$$p_i = c_i(1, \nu_i) \leq c_i(1, \lambda_i p_r) \leq c_i(1, \lambda_i p_r^*).$$

But then all these sectors must get automated (otherwise, their human wages ω_i must drift above $\lambda_i p_r^*$, a contradiction). That, in turn, means that robot demand must climb without bound. This in turn implies the robot sector must eventually get automated (otherwise, human wages in the robot sector will rise without limit, guaranteeing that at some point, $\omega_r > \lambda_r p_r^*$, where the finiteness of p_r^* is assured by the singularity condition and (20)).

And once the robot sector is automated, the robot price p_r hits p_r^* and remains there, regardless of the remainder of the price system. This is a special case of the non-substitution theorem in action: robot production has essentially been reduced to dated quantities of capital, which pins down the robot price relative to capital. It then follows that all final good sectors must get automated, by the same logic as used above: their prices are bounded above, so their demands and outputs must grow without limit, and the only way this can happen is if they get automated eventually.

It should also be noted that the education sector may or may not be automated as $\bar{K} \to \infty$. But this does not matter for the economy as a whole, as that sector shrinks relative to all other sectors combined. If the model is extended to accommodate potentially unbounded demand for education, then — under the singularity condition — the education sector will be automated as well.

Proposition 1 contains a converse, in case condition (17) fails. Now $c_r(1, \lambda_r p_r)$ never intersects p_r and the inequality in (19) always holds *strictly*; see Panel B of Figure 1. That in itself guarantees that the robot sector will never be automated. Now, as capital stocks rise, so will human wages, but that will also ensure that robot prices also keep pace with human compensation, both departing in tandem from the normalized price of physical capital. Moreover, it is now possible — though not inevitable — that several final good sectors will remain populated by human labor.

3.8. **The von Neumann Singularity and Factor Shares in National Income.** The singularity condition (17) has implications not just for long-run automation but also for the limiting shares of capital and human labor in national income. No other factor obtains a positive share, *including* robots or educations, because these sectors all produce under constant returns to scale. For instance, in any final goods sector, there will be a capital share, a human labor share, and a robot share of income produced. But the last item will flow to the robot industry, where the proceeds will be divided again among capital, humans and robots, leaving no surplus behind because of the zero-profit feature of a constant-returns-to-scale technology. The robot share (if any) in the robot sector again decomposes. Summing across such infinite chains, we obtain — in the end — only shares to physical capital (via the rental rate on capital) and to human labor (by aggregating across all wages and occupations) which must together exhaust national income.

In fact, when the singularity condition holds, even human labor is removed from consideration:

PROPOSITION **2.** *Suppose that the singularity condition* (17) *holds. As* $\bar{K} \to \infty$*, the share of physical capital in national income converges to 1, and that of human labor converges to zero.*

This Proposition captures the fundamental concern in this paper. Under the singularity condition (17), the functional share of physical capital must go to 1. It is the same concern that informs Piketty (2014), though in that work the presumed "central contradiction of capitalism" ($r > g$) is used to explain the capital share. (This is a well-known transversality condition, not a central contradiction, and besides, as is well-known, it cannot explain the rising share of capital.)

Our result is a statement about *functional*, not personal distribution, and it is embedded in two presumptions: (a) that physical capital is accumulated faster than human labor, and (b) that automation occurs sector by sector. A standard model without "micro-level" automation will not generate this result, as it would push us directly into the debate about the elasticity of substitution across capital and labor in the aggregate, which as already observed is an unsettled matter. With automation and the singularity condition, however, the rising share of capital is an inevitability. To harmonize our observations with the debates about the aggregate elasticity of substitution, we need an aggregation procedure that is firmly based on possible automation at the firm or industry level. In that aggregation, the overall elasticity of substitution is not a fixed parameter, but discipline would nevertheless be imposed by the structure of the model. Such an aggregation exercise (not conducted here) would form the basis for confronting our model with the data.

3.9. **Capital Accumulation and Wages.** Wage shares aside, we can also ask about the behavior of *absolute* wages. Despite ever-encroaching automation, it could remain sizable, and indeed, the overall wage level must rise with the extent of automation. We can guess where wages must go in the limit. Assume that the singularity condition holds. For every level of capital $\bar{K}$ the set of non-automated sectors is given by

$$\text{NA}(\bar{K}) \equiv \{j \in [0,1] \cup e \cup r | \omega_j > \lambda_j p_r\},$$

As capital continues to accumulate, the robot sector must become automated, by Proposition 1. Once that happens, the robot price settles at the value p_r^*, which is easily computed by reference to the robot sector alone; see (20). By virtue of the fact that all the other final goods sectors must get automated as well, the highest earned wage[7] must have a limit point of *at least* $\lambda^* p_r^*$, where $\lambda^* \equiv \max\{\lambda_r, \lambda_1\}$. On the other hand, the highest earned wage cannot have a limit point any larger than the largest of the three values $\lambda_1 p_r^*$, $\lambda_e p_r^*$, and $\lambda_r p_r^*$. After all, employment is strictly positive at every earned wage, and passing all three thresholds just described is tantamount to a zero demand for human labor.

Summarizing, the sequence of highest earned wages must have all its limit points in the interval $[\lambda^* p_r^*, \max\{\lambda^*, \lambda^e\} p_r^*]$, where p_r^* is defined by (20). If, in addition, there is free human mobility across sectors, then every sectoral wage is identical and the education sector is irrelevant, and so the limit of *the* wage rate in that case is given precisely by $\lambda^* p_r^*$. This value could be high. It could conceivably be infinite, if there is a sequence of sectors protected without bound: $\lambda_i \to \infty$ as $i \to 1$.[8] The fact that the labor share goes to zero, as stated in Proposition 2, certainly does not eliminate the possibility that *absolute* wages can be high.

Moreover, it is generally the case that wages must rise with the value of the capital stock, or at least with every wave of automation. That suggests that wages will rise monotonically over time even as labor is displaced. With some more restrictions on preferences and production technology to rule out perverse general equilibrium effects, this observation can be stated as a formal proposition.

In what follows, we assume that there is free human mobility across all sectors (where $e_{mj} = 0$ for all m, j). Let ω denote the common wage across all sectors. To cut down on different cases, assume that $\lambda_r \in (\lambda_0, \lambda_1)$, where we think of λ_0 as very small (or prone to automation) and λ_1 as very large (or highly protected). At any equilibrium, let ι stand for the equilibrium sector at the automation threshold: it is the smallest value of the index i such that

$$\lambda_i p^r = \omega, \tag{21}$$

so that it is clear that under the equilibrium in question, all sectors with $i < \iota$ will be automated, and all sectors with $i > \iota$ will be protected.

PROPOSITION **3.** *Assume that:*

(i) *Preferences and technology are both Cobb-Douglas, so that for every sector* j, $\alpha_j(\nu_j) = \alpha_j$ *and for every good* i, $\beta_i(\mathbf{p}) = \beta_i$.

(ii) *There is full mobility of labor across sectors (and so, in particular, the education sector is inactive.)*

Then:

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[7] Formally, define the *highest earned wage* to be the supremum over all human wages in sectors where positive employment occurs.

[8] We have eliminated this case in the formal analysis to handle technicalities regarding the continuity of demand, but of course we could make λ_1 arbitrarily high. In addition, with CES preferences, we can fully allow for this possibility by replacing $\lambda_1 < \infty$ with a suitable integrability condition on the collection $\{\lambda_i\}$.

(I) *Under* (i)*, the singularity condition* (17) *is automatically satisfied, and universal automation occurs as* $\bar{K} \to \infty$*.*

(II) *Capital accumulation strictly increases the automation threshold* ι *and the equilibrium wage rate* ω *provided that* $\lambda_\iota \neq \lambda_r$*. If* $\lambda_\iota = \lambda_r$*, the monotonicity is weak.*

(III) *At the same time, capital accumulation increases the share of capital in national income.*

This Proposition asserts that in the Cobb-Douglas case with free mobility of labor across sectors, automation of all sectors is inevitable: the singularity condition holds free of charge (Part I). Part II argues that as the capital stock increases, the equilibrium wage rate rises monotonically, Part III states that the share of capital in national income rises as well, at the same time that the real wage is rising. This brings out the implications of increased capital in the starkest possible way: functional inequality moves unambiguously in favor of capital while at the same time, real wages rise. With technical change, this co-movement could be reversed — real wages might fall with automation. Be that as it may, our baseline result without technical progress brings out the fundamental Ricardian drive underlying automation: that it is precisely the rising wage rate that contains within it the seeds of an ever-rising capital share.

4. PROOFS

Under the singularity condition (17), the following two lemmas establish a uniform upper bound on the price of capital relative to robots.

LEMMA **1.** *If* (17) *holds, then there is a unique value* $x^* \in (0, \infty)$ *such that*

(22) $$\lambda_r c_r(1, x) \gtreqless x \text{ for } x \lesseqgtr x^*, \text{ with equality when } x = x^*.$$

On the other hand, if (17) *fails, then*

(23) $$\lambda_r c_r(1, x) > x \text{ for all } x > 0.$$

Proof. Define the function $h(x) = \lambda_r c_r(1, x)$ for all $x > 0$. This is a continuous, increasing and concave function, and moreover,

$$\frac{h(x)}{x} = \lambda_r c_r\left(\frac{1}{x}, 1\right)$$

is strictly decreasing in x (after all, the right-hand side of this equation is strictly increasing in $1/x$). It is easy to verify that

(24) $$\lim_{x \to 0} \frac{h(x)}{x} > 1,$$

from our assumptions on f_r.[9] Moreover,

$$\lim_{x\to\infty}\frac{h(x)}{x}=\lim_{x\to\infty}\lambda_r c_r\left(\frac{1}{x},1\right)=\lim_{\rho\to 0}\lambda_r c_r\left(\rho,1\right). \tag{25}$$

Now, if condition (17) holds, it follows from (25) that $\lim_{x\to\infty}[h(x)/x]<1$, and combining this observation with (24) and the fact that $h(x)/x$ is strictly decreasing, we must conclude that there exists a unique value $x^*\in(0,\infty)$ such that (22). Otherwise, if (17) fails, then (25) implies that $\lim_{x\to\infty}[h(x)/x]\geq 1$, and once again combining this with (24) and $h(x)/x$ strictly decreasing, we have (23). ■

LEMMA **2.** *Under Condition* (17)*, there exists $p_r^*<\infty$, which depends only on λ_r and f_r, such that in any equilibrium $p_r\leq p_r^*$, with equality holding if the robot sector is automated.*

Proof. In any equilibrium, $p_r = c(1,\nu_r)\leq c(1,\lambda_r p_r)$, so that $\lambda_r p_r\leq\lambda_r c(1,\lambda_r p_r)$, with equality if the robot sector is automated. Using (22) of Lemma 1 applied to $x\equiv\lambda_r p_r$, we must conclude that $\lambda_r p_r\leq x^*$ in any equilibrium; again, with equality if the robot sector is automated. Define $p_r^*\equiv x^*/\lambda_r$ to complete the proof. ■

LEMMA **3.** *Under Condition* (17)*, the class of all equilibrium price functions (as $\bar{K}$ varies) is equicontinuous, and also uniformly bounded below by a strictly positive number.*

Proof. Under (17), Lemma 2 establishes an upper bound p_r^* for p_r. For any other sector j,

$$p_j=c_j(1,\nu_j)\leq c_j(1,\lambda_j p_r)\leq c_j(1,\lambda_j p_r^*)<\infty,$$

where p_r^* is given by Lemma (2). That establishes a uniform upper bound P on all price functions.

To obtain equicontinuity, we must show that for every $\epsilon>0$ there exists $\delta>0$ such that for any equilibrium price function and any pair of sectors i and j, whenever $|i-j|<\delta$, $p_i-p_j<\epsilon$. Because $p_j=c_j(1,\nu_j)$ for every sector j and because c_j is continuous on the compact set $[0,1]\cup e\cup r$ (and therefore uniformly continuous), it suffices to show the claimed property for the family of equilibrium labor price functions $\{\nu_j\}$.

Fix $\epsilon>0$; we define δ. First, because λ_i is continuous on $[0,1]$, it is uniformly continuous. Therefore there exists $\delta_1>0$ such that

$$P|\lambda_i-\lambda_j|<\epsilon \text{ whenever } |i-j|<\delta_1, \tag{26}$$

where P is the upper bound on all prices.

Next, recall the education function $e_{ii'}$. Define a function

$$\phi(i,j)\equiv\max_{m\in[0,1]}|Pe_{mi}-Pe_{mj}|.$$

[9] As $x\to 0$, there are two possibilities. Either capital input stays bounded away from zero for cost-minimizing production of one unit of r, or it converges to zero (perhaps along a subsequence of $x\to 0$). In the former case, obviously $\inf_{x>0}h(x)>0$, so we have (24). In the latter case, the cost-minimizing labor input $\ell(1,x)$ must go to infinity, otherwise we cannot stay on the unit isoquant. (That is because $f_r(0,\ell)=0$ and f_r is continuous.) But it is well known that the slope of $h(x)$ is given by $\lambda_r\ell_r(1,x)$. Therefore, the concave function $h(x)$ has $h'(0)=\infty$, which proves (24) in this second case.

Because $e_{ii'}$ is continuous, $\phi(i,j)$ is also continuous on $(i,j) \in [0,1]^2$ and therefore uniformly continuous on that domain. Therefore, for each $\epsilon > 0$, noting that $\phi(i,i) = 0$ for all i, there exists $\delta_2 > 0$ such that

$$\phi(i,j) < \epsilon \text{ whenever } |i-j| < \delta_2. \tag{27}$$

To complete the proof, define $\delta \equiv \min\{\delta_1, \delta_2\}$. Suppose that i and j are two final goods sectors with $|i-j| < \delta$. *Without loss of generality assume that* $\nu_i \geq \nu_j$.

If there are humans employed in sector j, then invoking condition (6),

$$\omega_i - \omega_j \leq p_e e_{mi} - p_e e_{mj} \leq \phi(i,j). \tag{28}$$

But $\phi(i,j) < \epsilon$ by (27) and $\delta \leq \delta_2$, so, using (28):

$$|\nu_i - \nu_j| = \nu_i - \nu_j \leq \omega_i - \omega_j \leq \phi(i,j) < \epsilon,$$

where the first weak inequality above invokes the fact that $\omega_j = \nu_j$ (by virtue of the fact that there are humans employed in sector j) and $\nu_i \leq \omega_i$. Otherwise, there are no humans in sector j, which implies that $\nu_j = \lambda_j p_r$. In this case, use (26) and $\delta \leq \delta_1$ to see that

$$|\nu_i - \nu_j| = \nu_i - \nu_j \leq p_r(\lambda_i - \lambda_j) \leq P|\lambda_i - \lambda_j| < \epsilon,$$

where the first weak inequality above uses $\omega_i \leq \lambda_i p_r$. The proof of equicontinuity is now complete.

The uniform lower bound on prices follows from the fact that for every sector j,

$$p_j = c_j(1, \nu_j) \geq c_j(1, \underline{\omega}) > 0$$

and the assumed continuity of j, which implies that $\inf_j c_j(1, \underline{\omega}) > 0$. Observe that p_r is bounded away from zero, since either the robot sector is automated in which case $p_r = p_r^*$, or else it is not whence $p_r = c_r(1, \omega_r) \geq c_r(1, \underline{\omega}) > 0$. Hence irrespective of whether or not sector j is automated, its price is bounded away from zero. ■

LEMMA **4.** *Under Condition* (17)*, the quantity demanded and produced of each final good must go to infinity with* $\bar{K}$*.*

Proof. We first claim that for each i, $\inf \beta_i(\mathbf{p}) > 0$, where the infimum is taken over every positive value of the capital stock, and every equilibrium price function corresponding to each capital stock. By Lemma 3, every equilibrium price function $\mathbf{p}$ is bounded and continuous, so by our assumption on preferences, $\beta_i(\mathbf{p}) > 0$. To complete the proof, let $\mathbf{p}^n$ be any sequence of equilibrium functions. By Lemma 3, the sequence $\mathbf{p}^n$ is equicontinuous, so by the Arzela-Ascoli theorem (see, for example, Dunford and Schwartz 1958, p. 382), there exists a bounded, continuous price function $\mathbf{p}^*$ and a subsequence n_k of n such that $\mathbf{p}^{n_k} \to \mathbf{p}^*$ in the sup norm. By Lemma 3 again, this function must be bounded below. By the assumed continuity of β_i in the sup norm, $\beta_i(\mathbf{p}^n) \to \beta_i(\mathbf{p}^*)$, but the latter is strictly positive by our assumption on preferences.

Furthermore, we know from (7) that $p_i y_i = \beta_i(\mathbf{p})[Y - p_e y_e]$, so that for every final good i,

$$y_i \geq \frac{\beta_i(\mathbf{p})[Y - p_e y_e]}{p_i} \geq \frac{d_i[Y - p_e y_e]}{a},$$

where d_i is the infimum of $\beta_i(\mathbf{p})$ over all capital stocks and equilibrium prices. Now, national income Y is given by (8), which implies that (in units of capital), $Y \geq \bar{K}$. Observe that p_e is bounded (above by $c_e(1, \lambda_e p_r^*)$) and so is y_e (since education costs are bounded). Therefore $y_i \to \infty$ as $\bar{K} \to \infty$. ■

LEMMA **5.** *Suppose Condition* (17) *holds. Consider the problem of unit cost minimization for any sector j:*

$$\min \hat{k} + \nu_j \hat{\ell}, \text{ subject to } f_j(\hat{k}, \hat{\ell}) \geq 1.$$

Then for every $\nu > 0$, there exists $\epsilon_j > 0$ such that if $\nu_j \leq \nu$, then $\hat{\ell} \geq \epsilon_j$.

Proof. Suppose not. Then there is a sequence of cost minimizing problems parameterized by ν_j^n such that the optimal unit labor use $\hat{\ell}^n \to 0$. Because both inputs are needed in production, it must be the case that $\hat{k}^n \to \infty$. But then $\hat{k}^n + \nu_j^n \hat{\ell}^n \to \infty$, which is a contradiction, because given Condition (17), a unit output can always be produced at bounded cost. ■

Proof of Proposition 1. Assume (17). Then by Lemma 2, $p_r \leq p_r^*$, so that for every sector $i \in [0, 1]$, $\nu_j \leq \lambda_j p_r \leq \lambda_j p_r^* < \infty$. By Lemma 5, there exists $\epsilon_j > 0$ such that unit labor use $\hat{\ell}_j \geq \epsilon_j$. By Lemma 4, the output of every final good must go to ∞. It follows that *aggregate* labor use in that sector must grow unboundedly large: $\ell_i \to \infty$ as $\bar{K} \to \infty$. Because the aggregate human labor endowment of the economy is bounded, this implies that almost every final goods sector must be automated as $\bar{K} \to \infty$.

By Lemma 4 and the results just established, the demand for robots must grow without bound. Therefore the robot sector must eventually get automated. The proof uses exactly the same argument applies as the one made for final goods sectors.

Suppose now that (17) fails. Then, by (23) of Lemma 1, we know that $\lambda_r c_r(1, x) > x$ for all $x > 0$. Setting $x = \lambda_r p_r$, we must conclude that

$$p_r < c_r(1, \lambda_r p_r)$$

in any equilibrium. But we know that $p_r = c_r(1, \nu_r)$. It follows that

$$\lambda_r p_r > \nu_r, \tag{29}$$

so that in any equilibrium, the robot sector is protected. Finally, if there is free movement of labor, then for every sector j, $\omega_j = \omega_r$. Using (29), we must conclude that whenever $\lambda_j \geq \lambda_r$,

$$\lambda_j p_r \geq \lambda_r p_r > \nu_r = \omega_r = \omega_j,$$

so that such sectors must be protected as well, irrespective of the value of $\bar{K}$. ■

Proof of Proposition 2. Assume that (17) holds. Begin by recalling the market clearing condition (16) for physical capital:

$$\bar{K} = \int_0^1 [1 - \alpha_i(\nu_i)] p_i y_i di + [1 - \alpha_r(\nu_r)] p_r y_r + [1 - \alpha_e(\nu_e)] p_e y_e.$$

Divide by national income Y on both sides and use equation (8) to obtain

$$\frac{\bar{K}}{Y} = \frac{Y - p_e y_e}{Y} \int_0^1 [1 - \alpha_i(\nu_i)]\beta_i(\mathbf{p})di + [1 - \alpha_r(\nu_r)]\frac{p_r y_r}{Y} + [1 - \alpha_e(\nu_e)]\frac{p_e y_e}{Y}. \tag{30}$$

By Proposition 1, the robot sector will be eventually automated (for large enough values of $\bar{K}$). If that has happened for some $\bar{K}$ and $A(\bar{K})$ is the set of automated final goods sectors under $\bar{K}$, then

$$\frac{p_r y_r}{Y} \geq \frac{Y - p_e y_e}{Y} \int_{i \in A(\bar{K})} \alpha_i(\nu_i)\beta_i(\mathbf{p})di + \alpha_r(\nu_r)\frac{p_r y_r}{Y}, \tag{31}$$

where the inequality occurs because we cannot be sure that the education sector is automated. Combine equations (30) and (31) to get

$$\frac{\bar{K}}{Y} = 1 - \frac{Y - p_e y_e}{Y} \int_{i \in A(\bar{K})^c} \alpha_i(\nu_i)\beta_i(\mathbf{p})di - \alpha_e(\nu_e)\frac{p_e y_e}{Y}. \tag{32}$$

Now pass to the limit as $\bar{K}$ (and Y) goes to infinity. Notice that $p_e y_e$ is bounded, because p_e is bounded (an implication of Lemma 3) while $y_e \leq \max_{j,j'} e_{j,j'} < \infty$ by the continuity of $e_{jj'}$ in j and j'. Moreover, because almost every final goods sector is automated in the limit (Proposition 1), the measure of $A(\bar{K})^c$ converges to zero. It follows from these observations and (32) that

$$\lim_{\bar{K} \to \infty} \frac{\bar{K}}{Y} = 1.$$

which completes the proof. ∎

Proof of Proposition 3. It is easy to verify (I) by recalling (17) in the CES case; see (18). In the Cobb-Douglas case, $\sigma = 1$ and the condition is automatically satisfied. Because the education sector is inactive, Proposition 1 implies the automation of every sector in the limit.

To prove (II), note that the capital market clearing condition (16) implies that

$$\begin{aligned} \bar{K} &= \int_0^1 (1 - \alpha_i)p_i y_i di + (1 - \alpha_r)p_r y_r = Y \left[\int_0^1 (1 - \alpha_i)\beta_i di\right] + (1 - \alpha_r)p_r y_r \\ &= (\bar{K} + \omega)\left[\int_0^1 (1 - \alpha_i)\beta_i di\right] + (1 - \alpha_r)p_r y_r, \end{aligned} \tag{33}$$

where the third equality uses (7) and free mobility to obtain $Y = \bar{K} + \omega$. At the same time,

$$p_r y_r = Y \int_0^\iota \alpha_i \beta_i di + \gamma \alpha_r p_r y_r, \tag{34}$$

where γ is the fraction of the value of labor demand in the robot sector filled by robots. If $\lambda_\iota < \lambda_r$, this fraction is 0 and the robot sector is not automated. If $\lambda_\iota > \lambda_r$, the robot sector is fully automated and $\gamma = 1$, while if $\lambda_\iota = \lambda_r$, $\gamma \in [0, 1]$. Combining (33) and (34), we see that

$$\frac{\bar{K}}{\bar{K} + \omega} = \int_0^1 (1 - \alpha_i)\beta_i di + \frac{1 - \alpha_r}{1 - \gamma\alpha_r} \int_0^\iota \alpha_i \beta_i di. \tag{35}$$

Let $\lambda_\iota \neq \lambda_r$. Suppose on the contrary that $\bar{K}$ increases while ι declines or stays the same. It follows from (21) that ω/p^r cannot rise. Now $p_r = c_r(1, \omega)$, which means that

$$\frac{p_r}{\omega} = c_r\left(\frac{1}{\omega}, 1\right).$$

Because p_r/ω cannot fall, this equation implies that ω cannot rise as $\bar{K}$ goes up. Using this information in (35), we must conclude that the left-hand side goes up (strictly). The right-hand side weakly falls on account of ι. Moreover, because ι has declined, γ can at best fall or stay constant. Combining these two effects, we see that the right-hand side weakly declines. This mean that (35) cannot hold after the rise in $\bar{K}$, a contradiction.

If $\lambda_\iota = \lambda_r$, then the same argument can be used to show that neither ι nor ω can strictly decline after an increase in $\bar{K}$.

To prove (III), suppose that $\lambda_\iota \neq \lambda_r$. Then we have shown that ι strictly increases with $\bar{K}$, and for this reason it is impossible for γ to decline. Therefore, by (35), the share of capital in national income must rise. On the other hand if $\lambda_\iota = \lambda_r$, then the resulting monotonicity of ι is weak. If ι strictly increases nevertheless, the previous argument applies, but if not, then ω must be constant as well. In this case, $\bar{K}/(\bar{K} + \omega)$ trivially increases in $\bar{K}$, and the proof is complete. ■

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