Search Frictions and Gendered Skill Investment in Marriage Markets *

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Abstract

I study a decentralized marriage market with search frictions, costly skill investments, and non-transferable utility. Agents differ in their costs of acquiring high skills and engage in search for a potential match. Payoffs depend on the skill levels of both partners. Despite a symmetric underlying environment—where neither payoffs nor the matching technology favor a gender—the market can exhibit asymmetric equilibria. A larger fraction of agents from one side invests in skills, while fewer from the other side do. High skill premiums drive these asymmetries, which can uniquely arise under specific conditions. A microfounded household labor supply model links rising wages for high-skilled individuals to these outcomes, offering insights into declining female labor force participation.

Keywords: marriage market, costly investment, female labor force participation

JEL Codes:

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1. Introduction

I study a decentralized matching market with three features: First, participants face search frictions; second, participants make costly investment decisions to acquire skills; and third, transfers are not available to equilibrate matches, i.e., payoffs are non-transferable. The paper makes two contributions. First, I develop a simple and tractable model to examine skill investment decisions within a search-and-matching framework under non-transferable utility. Second, I demonstrate how a rise in skill premium—where members of a household receive higher payoffs from having at least one highly skilled member in the household—can create gender-based disparities in skill acquisition. Specifically, I show that rising skill premiums can lead to asymmetries in how males and females invest in skills, which may reflect patterns of discrimination.

The motivation to explore whether gender-based disparities can emerge as equilibrium outcomes stems from the documented decline in female labor force participation in several developing countries (see Klasen, 2019, for instance). This trend has coincided with a rapid rise in high-skill sectors like information technology in India, where the returns to being highly skilled have grown significantly. How might this increased skill premium affect pre-marital investments in skills?

On one hand, higher skill premiums could incentivize greater investments in education to enhance labor market outcomes. On the other hand, there is a countervailing force: if the marginal value of having a highly-skilled spouse decreases when one is highly skilled themselves, individuals with moderate talent—those for whom skill acquisition is costly—might avoid these investments. Instead, they may rely on matching with highly-skilled partners, anticipating household specialization. In such cases, the highly-skilled partner focuses on labor market participation while the low-skilled partner assumes greater responsibility for domestic production. Such specialization can exacerbate disparities: as more individuals on one side of the market invest in skills, a substitution effect might occur, discouraging skill acquisition on the other side.

This feedback loop is at the core of this paper. The main results demonstrate how changes in the underlying payoff structure—where household payoffs increase significantly when at least one member is highly skilled—can create gender-based disparities in skill acquisition, despite a fully symmetric underlying environment. Crucially, with a low skill premium, the unique equilibrium is symmetric: an identical proportion of men and women invest in skill acquisition. However, when the skill premium increases, we can have a unique equilibrium that is asymmetric: a far greater proportion of one gender invests in acquiring skills, while fewer people from the other gender acquire skills. Additionally, through a parameterized example, I provide a microfounded model of a non-cooperative game within a household in the spirit of Chiappori (1988) to illustrate how such changes in payoffs may arise and show how this mechanism can drive a decline in labor force participation on one side of the market.

I now briefly describe the framework. The market consists of a continuum of agents from two sides, men and women. Agents engage in an undirected search to find a potential match. Prior to beginning their search, each decides whether to incur a cost to acquire high skills or to remain low skilled. An agent's cost depends on their type, indexed by their identity in the unit interval. An agent with a higher type incurs a higher cost to acquire high skills. After making this investment choice, they begin the search phase wherein they meet partners from the other side randomly. When two people meet, they observe each other's skills and decide whether to match with each other. Their payoffs conditional on matching depends only on their skills and not their latent type. Therefore, there are four possible payoffs for an agent conditional on matching, denoted by $\phi(H, H), \phi(H, L), \phi(L, H), \phi(L, L)$, where the first argument is their own skills and the second is their partners' skill level. If two agents match, they immediately leave the market and are replaced by their identical copies. Otherwise, they continue searching. This simple construction ensures that the economy is perpetually in the steady state. Our goal is to understand the structure of pure strategy equilibria and their relation with the payoff function $\phi(\cdot, \cdot)$. In particular, I seek to understand the factors that lead to equilibria with asymmetric levels of investment from the two sides of the market. The point of the exercise, then, is to illustrate that disparities can arise even when the underlying environment is fully symmetric in terms of preferences, payoffs, and the matching technology.

I briefly digress to highlight the three key features of the model. Among these, search frictions and costly investments are particularly pervasive. Individuals—or their parents in some contexts—make premarital investments anticipating labor market outcomes that influence their marriage prospects. Meanwhile, the prevalence of dating apps and marriage bureaus serves as clear evidence of search frictions in modern marriage markets. The third feature of non-transferable utility (NTU) is motivated by social matches such as marriages. The seminal paper by Becker (1973) also briefly considered such an environment. To quote Smith (2006) (who also studied a marriage market with search frictions and NTU),

> *But in defense of the NTU model, disagreements about matching in social settings are not uncommon. And whenever we observe a potential match or split desired by one party but not the other, utility is obviously not fully transferable. For the total match surplus either is positive or is not, and there can be no disagreement.*

A rather pronounced case of such a disagreement is a sharp discontinuity around $\frac{1}{2}$ in the share of income earned by the wife in couples documented by Bertrand et al. (2015). They show that, in the context of couples in the US, there is considerable aversion to a situation where the wife earns more than her husband. A world with fully transferable would show no such discontinuity: whatever may be the psychological cost to the husband due to wife earning more, can be compensated through transfers.

We aim to characterize the equilibrium structure, with a particular focus on agents' investment decisions. Proposition 1 establishes existence, allowing us to turn to the nature of equilibrium outcomes. To do so, we first analyze the post-investment matching game. I assume that every agent strictly prefers a highly skilled partner over a low-skilled one. This immediately rules out negative assortative matching, as a high-skilled agent will never reject another high-skilled agent. Thus, equilibria must take one of two forms: either a positive assortative matching (PAM) equilibrium, where high-skilled agents match exclusively with high-skilled partners, or an all-match (AM) equilibrium, where all agents accept any available partner.[1]

As expected, the ability to reject low-skilled partners in favor of high-skilled ones depends on market conditions. If the market is too thin—meaning agents face long waiting times before meeting a partner—rejecting a match becomes costly, making PAM less viable. Proposition 2 formalizes this by showing that PAM exists only if the market is sufficiently thick and the marginal gains of a high-skilled match are large enough.

From an investment perspective, PAM induces symmetric investment patterns: both men and women follow a common cutoff rule—agents below a threshold invest in high skills, while those above it do not. The intuition is straightforward. Under PAM, investments act as strategic complements—as more women become highly skilled and begin rejecting low-skilled men, marginal men are further incentivized to invest in skills to secure a high-skilled match.

Therefore, if at all there is to be an asymmetric equilibrium, it cannot involve a PAM as a continuation equilibrium. However, even if the matching stage follows an AM equilibrium, Proposition 3 shows that investment decisions remain symmetric if the payoff function is supermodular. This follows naturally—supermodularity fosters complementarities in investment, reinforcing symmetric skill acquisition.[2]

What happens if the payoff function is submodular? In sufficiently thin markets, PAM is ruled out, as discussed earlier. Within these environments, symmetric equilibria—when they exist—are unique among symmetric strategies. However, Proposition 4 establishes the possibility of asymmetric equilibria, where one side of the market entirely forgoes skill investment. To obtain sharper results, we further specialize to affine cost functions, assuming that the cost of acquiring high skills varies affinely with type.

Proposition 5 fully characterizes this environment—moderately thin markets with submodular payoffs and affine costs. The key takeaway is that only three

[1]In some cases, we could also envision other asymmetric equilibria where the two sides invest asymmetrically in skill acquisition and, subsequently, one side of the market rejects all the low-skill partners from the other side. However, those can be ruled with additional assumptions.

[2]For instance, Atakan et al. (2024) obtain PAM in a search environment, albeit in a TU framework.

types of equilibria are possible.

1. Symmetric Equilibrium: A cutoff type x_{sym} exists, below which men and women invest in skills.
2. Full Investment from One Side (FIOS): All men acquire skills, while only women below a type $\underline{x}$ do.[3]
3. No Investment from One Side (NIOS): No men invest in skills, while women below a type $\bar{x}$ do.

As the notation suggests, these thresholds satisfy $\bar{x} \geqslant x_{sym} \geqslant \underline{x}$. Importantly, FIOS and NIOS equilibria can never coexist. Moreover, asymmetric equilibria are not only possible but, in some cases, unique (among pure strategy equilibria).

It is worth noting that several papers have documented some disparities arising out of structural asymmetries such as fertility concerns Low (2024); Siow (1998); Zhang (2021). In contrast, disparities in skill acquisition between men and women can arise in equilibrium despite the absence of any asymmetry in the payoff or matching technology. This disparity arises due to search frictions and NTU.[4] At its core though, the underlying intuition is very much like one from public goods games. Essentially, high-skills are somewhat like a public good. When a person acquires high skills, even their partner benefits. But the cost of acquiring this high-skill is borne by the person acquiring the skills. Thus, we can have equilibria where one side invests asymmetrically more in providing this public good. This reasoning, although somewhat on the right lines, is incomplete because of two reasons. First, agents' payoffs are increasing in their skills even if the partner is highly-skilled. Thus, there is some incentive to acquire high skills regardless of the spouse's skill level.[5] Second, even in the case of submodular payoffs where this public nature appears to be the strongest, we have environments that admit only the symmetric equilibrium.

How does this asymmetry relate to increasing skill premium that I discussed at the start of the introduction though? This question is answered in Proposition 6. Essentially, given any environment that only admits the symmetric equilibrium, if $\phi(H, H) - \phi(L, L)$, the difference between the highest and the lowest payoff, is sufficiently low, then we can construct another environment with the following two features: (i) $\phi(L, L)$ remains unchanged, and (ii) the payoff to every member of the household with at least one highly-skilled person is higher, such that the new environment admits a unique equilibrium, the FIOS one. Therefore, not only

[3]Naturally, the gender assignment is arbitrary—an analogous equilibrium exists where all women acquire skills and men below $\underline{x}$ do.

[4]If we had no search frictions and were in the TU world, then a submodular payoff function leads to a unique stable matching that is negatively assortative. I conjecture that this would lead to asymmetry in skill acquisition but I do not have a reference to this effect.

[5]For instance, if I started with the assumption that $\phi(H, H) < \phi(H, L)$, i.e., if one is high-skilled then they prefer to be matched with a low-skilled person, then obtaining this asymmetry would be straightforward in some cases. Essentially, it would be optimal for one side to invest in skills and the other side not to in this case.

can asymmetry may arise in equilibrium, but rather, in some environments, it will necessarily arise.

As a final step, I even provide a parametric example that numerically illustrates that the changes in the payoff functions that give rise to asymmetric equilibria can be easily microfounded. This approach is atypical in the matching literature where the payoff functions from matching are taken as exogenous. However, I write a simple non-cooperative game between the husband and the wife following Chiappori (1988).[6] The utility of each member has a public component that depends on the total household income and domestic production. Members decide how to split their time working outside and towards domestic production. Working outside generates income proportional to their wages which depend on their skills. Domestic production is an increasing function of the total effort expended on it. Members view domestic production as a costly activity (relative to working outside), and the costs are private. Example 1 demonstrates how increasing wages in such a setup can lead us from a situation where we have from a unique symmetric equilibrium to two equilibria, one symmetric and one NIOS.

I end this section with one remark. Technically, the NTU assumption offers a considerable simplicity to the problem. The reason is that, in the TU framework, whenever two agents match we only know their *joint surplus.* But their individual values—which determine their incentive constraints—have to be determined endogenously. In contrast, the exogeneity of the payoff enables us to cleanly characterize the equilibrium values and agents' incentives. Finally, this feature also enables the comparative static on skill premium which is the main point of the paper.

2. Related Literature

Naturally, this paper is related to the large literature on matching markets. There are two broad directions in which this literature has evolved. The initial papers, such as by Becker (1973); Gale and Shapley (1962); Shapley and Shubik (1971) assumed a frictionless matching markets. Somewhat later, Shimer and Smith (2000, 2001); Smith (2006) investigated these markets with search frictions. Across these two frameworks, another important distinction is between transferable utility (such as Becker (1973); Shapley and Shubik (1971)) vs non-transferable utility (such as Gale and Shapley (1962); Smith (2006)). The post-investment stage of my game can be viewed as a decentralized version of Gale and Shapley (1962) which was studied by Adachi (2003); Smith (2006). To this setting, I add an investment choice before the search phase. A large literature has studied costly investments in marriage markets, (see Bhaskar and Hopkins, 2016; Bhaskar et al., 2023; Chade and Lindenlaub, 2022; Cole et al., 2001; Mailath et al., 2013; Nöldeke

[6] See Chiappori (2020) for a detailed survey on various models of household labor supply decisions as games.

and Samuelson, 2015; Peters, 2007, for instance). Most of these papers study the efficiency of such investments. With the exception of Chade and Lindenlaub (2022) and Bhaskar and Hopkins (2016) most other papers assume that the returns are deterministic, while these two study various issues related to how riskiness of investments affects the match outcomes as well as investments themselves. Also, Bhaskar et al. (2023) highlight how asymmetries in investments can occur when the two genders differ in their bargaining power. Crucially, in contrast to my paper, all of these papers have frictionless matching markets. Recently, Atakan et al. (2024) study a marriage market with search frictions, costly investments, and *transferable utility*. Their main finding is that the outcomes are constrained efficient and assortative in a wide variety of settings. In contrast, adopting the NTU framework, I sidestep the question of effiiency and focus solely on the disparities in skill acquisition that arise in equilibrium.

Given my emphasis on the gender differences in skill acquisition, I am naturally related to the papers that highlight these issues in various contexts, e.g. Chiappori et al. (2009, 2017); Low (2024); Zhang (2021). Often, unlike the current paper, these papers assume some innate differences in preferences or the technological environment that give rise to these disparities.

3. Model

The economy consists of a unit mass of men and a unit mass of women, each indexed by a type $x \in [0, 1]$. Upon "birth," agents choose a skill level from a finite set $\mathcal{S}$, which is assumed to be binary for most of the paper: $\mathcal{S} = \{H, L\}$, where H represents high skill and L represents low skill. The cost of acquiring skill s for an agent of type x is $C(x, s)$. Normalizing $C(x, L) = 0$, we denote $C(x, H)$ simply as $C(x)$, assuming it is increasing and weakly convex—higher-skilled individuals face higher costs.

If a man with skill s and a woman with skill s' match, the man's payoff is $\phi(s, s')$, and the woman's is $\phi(s', s)$. That is, the first argument represents an individual's own skill and the second, their partner's. This is a non-transferable utility (NTU) setting where payoffs depend only on skill levels, not on underlying type. Agents discount future payoffs at rate $r > 0$.

We impose the following standard ranking of payoffs:

Assumption 1: $\phi(H, H) \geqslant \phi(H, L) \geqslant \phi(L, H) \geqslant \phi(L, L)$.

This assumption can be microfounded through a household decision-making problem as in Chiappori (1988). See Section 6 for an example.

An economy is a tuple $\langle \phi, C, \lambda, r \rangle$ consisting of the payoff functions, the cost functions for acquiring different skill levels, and the arrival rate of potential partners that we will describe momentarily.

Timing: Time is continuous and infinite. At each instant, unmatched agents engage in costless, undirected search, observing the skill distribution of the opposite pool. When two agents meet, they observe each other's skills and decide whether to match. If they do, they leave the market and are replaced by identical unmatched copies. Newly born agents make an irreversible skill investment decision before entering the unmatched pool.

Thus, the economy is perpetually in "steady state", i.e., the mass of each set of agents, men and women, remains to be 1 and their type distribution remains unchanged. Unmatched agents meet others drawn at random from the unmatched pool at an exponential rate λ.[7]

Strategies: We restrict attention to stationary equilibria (where strategies of any type do not vary with time). A strategy for a type x and gender γ specifies a choice of skill to acquire, and then, subsequently, an acceptance set during the search phase. This specifies the set of agents one is willing to match with should they meet them. It is without loss to restrict attention to pure strategies insofar as investment is concerned. We denote by $I^\gamma : [0,1] \to \{0,1\}$ the investment strategy of a type x agent of gender γ, where one indicates their decision to acquire high skills and zero does not. We assume that I^γ is measurable. Given players' investment strategies, let $\zeta_I^\gamma := \int I^\gamma(x)\mathrm{d}x$ denote the proportion of H types in group γ. Since the dependence on I is obvious in most cases, we will omit it. Post the agents' investment decisions, they engage in an undirected search. Whenever they meet someone, they have to decide whether to match with that person or not. If both the agents agree to a match, then the match takes place. The acceptance strategy specifies the probability of accepting an agent of each skill type and depends on one's own type as well as the proportion of the high types from the opposite pool. First, given Assumption 1, no agent would ever turn down a high type agent. Therefore, we merely need to specify the probability of accepting a low type for each gender and skill type. We denote by $\alpha^\gamma : [0,1] \times \{H,L\} \to [0,1]$ the probability that an agent of gender γ and skill s accepts a low type agent of the opposite gender given the proportion of the high types from group γ'. Often, this dependence on $\zeta^{\gamma'}$ is obvious and, therefore, we will suppress it.

3.1. Equilibrium

Agents choose their skill levels optimally given their expected value minus the cost. Let us first write each skill types' expected value. For $\gamma, \gamma' \in \{M,W\}$ such

[7] Since the economy is perpetually in the steady state with a unit mass of population on either side, we do not need to scale it with the population size.

that $\gamma \neq \gamma'$ and given players' strategies $(I^\gamma, \alpha^\gamma)_{\gamma \in \{m,w\}}$, we have

$$V^\gamma(H, \alpha^\gamma; \zeta^{\gamma'}, \alpha^{\gamma'}) := \frac{\lambda\left[\zeta^{\gamma'}\phi(H,H) + (1-\zeta^{\gamma'})\alpha^\gamma(H)\phi(H,L)\right]}{r + \lambda(\zeta^{\gamma'} + (1-\zeta^{\gamma'})\alpha^\gamma(H))} \tag{Values}$$

$$V^\gamma(L, \alpha^\gamma; \zeta^{\gamma'}, \alpha^{\gamma'}) := \frac{\lambda\left[\zeta^{\gamma'}\alpha^{\gamma'}(H)\phi(L,H) + (1-\zeta^{\gamma'})\alpha^\gamma(L)\alpha^{\gamma'}(L)\phi(L,L)\right]}{r + \lambda(\zeta^{\gamma'}\alpha^{\gamma'}(H) + (1-\zeta^{\gamma'})\alpha^\gamma(L)\alpha^{\gamma'}(L))}$$

These values can be obtained through straightforward recursive equations that we omit. An agent of skill s and gender γ chooses to maximize $V(s, \alpha^\gamma; \zeta^{\gamma'}, \alpha^{\gamma'})$ given $(\zeta^{\gamma'}, \alpha^{\gamma'})$.

An equilibrium consists of the investment and acceptance strategies for each gender and skill type, $\langle (I^\gamma, \alpha^\gamma)_{\gamma \in \{M,W\}} \rangle$, satisfying (IC-invest) and (IC-AL) below.

$$V^\gamma(H, \alpha^\gamma; \zeta^{\gamma'}, \alpha^{\gamma'}) - C(x) - V^\gamma(L, \alpha^\gamma; \zeta^{\gamma'}, \alpha^{\gamma'}) \begin{cases} > 0 \implies I^\gamma(x) = 1 \\ = 0 \implies I^\gamma(x) \in \{0,1\} \\ < 0 \implies I^\gamma(x) = 0 \end{cases} \tag{IC-invest}$$

$$V^\gamma(s, \alpha^\gamma; \zeta^{\gamma'}, \alpha^{\gamma'}) \begin{cases} < \phi(s,L) \implies \alpha^\gamma(s) = 1 \\ = \phi(s,L) \implies \alpha^\gamma(s) \in [0,1] \\ > \phi(s,L) \implies \alpha^\gamma(s) = 0 \end{cases} \tag{IC-AL}$$

(IC-invest) ensures that the agents' investment decisions are optimal, while (IC-AL) specifies that agents acceptance decisions are optimal. In particular, if the value to agent is higher than what (s)he would receive by matching with a low-skilled person, then (s)he must not be accepting the low-skilled person. As the proof of Proposition 1 shows, in any equilbrium, the investment strategies are characterized by a cutoff type, $\hat{x}_\gamma$ for each γ, such that an agent i of gender γ and type x_i invests if and only if $x_i \leqslant \hat{x}_\gamma$.

Notation 1: *In light of the above, we will denote an equilibrium by $(\hat{X}, \alpha)$ where each, $\hat{X} = (\hat{x}_m, \hat{x}_w)$ and $\alpha = (\alpha^\gamma(H), \alpha^\gamma(L))_{\gamma \in \{m,w\}}$ are vectors denoting the relevant strategies. In general, X denotes a vector (x_m, x_w).*

To avoid getting sidetracked, let us first establish that an equilibrium exists.

Proposition 1: *An equilibrium always exists.*

Having taken care of the existence issue, we turn to the more substantive issues. What type of sorting do we see in the equilibria, and more importantly, what are the investments that agents make? In light of Assumption 1, no agent will ever reject a high skilled person. Therefore, we can rule out the existence of a negatively assortative match. Thus, insofar as pure strategy equilibria are concerned, we are left with the following two possibilities:

1. All match (AM): Agents match with the first agent they meet.
2. Positive Assortative Matching (PAM): High types match with high types, and low types match with low types.

There is also a third possible candidate wherein one side of the market only accepts the high-skill partners from the other side and the low skilled people remain permanently unmatched. Such equilibria have to be necessarily asymmetric if they exist. Furthermore, those can be ruled with additional assumptions. Regardless, I focus on equilibria where every agent is eventually matched with probability one.

Of course, understanding the structure of equilibria *given the initial investments* is not particularly challenging. The object of our interest though is, primarily, the nature of investments that arise in equilibria and then, subsequently, the structure of the continuation equilibria. Towards this goal, let us understand the incentive constraints for each of the above equilibria.

3.2. AM

In the case of this match, $\alpha^\gamma(\cdot) = 1$ for all γ. Given $\widehat{X}$, the players' payoffs (with some abuse of notation), denoted by $V_A^\gamma(\cdot)$ are:

$$V_A^\gamma(H;\widehat{X}) = \frac{\lambda[\widehat{x}_{\gamma'}\phi(H,H) + (1-\widehat{x}_{\gamma'})\phi(H,L)]}{r+\lambda}, \quad \text{and}$$
$$V_A^\gamma(L;\widehat{X}) = \frac{\lambda[\widehat{x}_{\gamma'}\phi(L,H) + (1-\widehat{x}_{\gamma'})\phi(L,L)]}{r+\lambda}.$$

For $(\widehat{X},\alpha)$ as specified to be an equilibrium, we need that α be an equilibrium in the continuation game given $\widehat{X}$ and that $\widehat{X}$ be optimal given the selected equilibrium in the continuation game. Let us ignore the trivial cases where either everybody invests or nobody does, and assume that at least one of $\widehat{x}_m, \widehat{x}_w$ is interior. In the case, the IC constraints for this equilibrium are:

$$V_A^\gamma(H;\widehat{X}) - C(\widehat{x}_\gamma) = V_A^\gamma(L;\widehat{X}) \quad \text{if } \widehat{x}_\gamma \in (0,1) \qquad \text{(AM: Invest)}$$
$$V_A^\gamma(H;\widehat{X}) \leqslant \phi(H,L) \qquad \text{(AM: H-L)}$$
$$V_A^\gamma(L;\widehat{X}) \leqslant \phi(L,L) \qquad \text{(AM: L-L)}$$

(AM: Invest) ensures that $\widehat{x}_\gamma$ is indifferent between investing and not if $\widehat{x}_\gamma \in (0,1)$. (AM: H-L) and (AM: L-L) guaranty that an H or L type respectively do indeed match with an L type should they meet with each other.

4. Analysis of the PAM equilibria

Here, $\alpha^\gamma(H) = 0, \alpha^\gamma(L) = 1$. So, the players' payoffs (with some abuse of notation), denoted by $V_P^\gamma(\cdot)$, are:

$$V_P^\gamma(H; \widehat{X}) = \frac{\lambda \widehat{x}_{\gamma'} \phi(H,H)}{r + \lambda \widehat{x}_{\gamma'}} \quad \text{and} \quad V_P^\gamma(L; \widehat{X}) = \frac{\lambda(1-\widehat{x}_{\gamma'})\phi(L,L)}{r + \lambda(1-\widehat{x}_{\gamma'})} \tag{PAM: Values}$$

And for such an equilibrium to exist with an interior $\widehat{X}$, the IC constraints are:

$$V_P^\gamma(H; \widehat{X}) - C(\widehat{x}_\gamma) = V_P^\gamma(L; \widehat{X}) \tag{PAM: Invest}$$

$$V_P^\gamma(H; \widehat{X}) \geqslant \phi(H,L) \tag{PAM: Assortativity}$$

(PAM: Invest) checks for the indifference of the cutoff type $\widehat{x}_\gamma$, while (PAM: Assortativity) ensures that a high-skill person prefers to wait to match with another high-skill person that matching with a low-skill person if they meet one.

Proposition 2: *No PAM exists if $\lambda[\phi(H,H) - \phi(H,L)] < r\phi(H,L)$. Whenever a PAM exists it is symmetric, i.e., $\widehat{x}_m = \widehat{x}_w$.*

Since a PAM is always symmetric, if the market witnesses ex-post asymmetry in skill acquisition from the two sides, it must necessarily come from an AM equilibrium. Before we analyze those further, I will remark that it is straightforward to construct parameters so that a PAM exists. Thus, Proposition 2 is not vacuous. Moreover, the sufficient condition in the non-existence of a PAM trades off the marginal value of meeting a high-skilled partner ($\lambda[\phi(H,H) - \phi(H,L)]$) with the foregone benefits of matching a low-skilled partner that is currently available ($r\phi(H,L)$). Whenever the marginal value of waiting to meet an H type is sufficiently low (even if the entire population of the other side is highly skilled), an agent would not find it worthwhile to let go of an L type whom (s)he meets. In this regard, let us focus on the cases where no PAM exists and analyze the AM equilibria in greater depth.

5. Analysis of the AM equilibria

Let us define two objects that appear repeatedly in the proofs of the results that follow.

$$\Delta := \phi(H,H) + \phi(L,L) - \phi(H,L) - \phi(L,H)$$
$$\Delta_h := \phi(H,L) - \phi(L,L).$$

By Assumption 1, $\Delta_h \geqslant 0$. Moreover, $\Delta \geqslant (\leqslant) 0$ if $\phi(\cdot,\cdot)$ is supermodular (submodular).

Proposition 3: *If $\phi(\cdot,\cdot)$ is supermodular, then all AM equilibria are symmetric, i.e., $\widehat{x}_m = \widehat{x}_w$.*

Therefore, if there is any ex post asymmetry in investments between the two sides, it cannot arise in a PAM equilibrium or in an AM equilibrium if $\phi(\cdot,\cdot)$ is supermodular. Thus, if at all there is a possibility of some asymmetry, it must come from a submodular $\phi(\cdot,\cdot)$ function, to which we now turn. Also, since the underlying environment is symmetric, if there is an asymmetric equilibrium $(\widehat{x}_m, \widehat{x}_w)$, then there is also an asymmetric equilibrium $(\widehat{x}_w, \widehat{x}_m)$. Therefore, when we say that there is a unique asymmetric equilibrium it would typically mean up to permutation.

Proposition 4: *Suppose that $\phi(\cdot,\cdot)$ is submodular. Then, the following hold.*

1. *$\exists$ a symmetric equilibrium if $\lambda[\phi(H,H)-\phi(H,L)] \leqslant r\phi(H,L)$ and $\lambda[\phi(L,H)-\phi(L,L)] \leqslant r\phi(L,L)$. Whenever a symmetric equilibrium exists, it is unique.*
2. *If $|\Delta|$ is sufficiently large, then there is a unique asymmetric equilibrium (up to permutation), $(0, x^*)$, wherein women with type below x^* acquire high skills while no man does.*

The first part of Proposition 4 establishes the existence of a symmetric AM equilibrium under two conditions, one of which guarantees non-existence of a PAM equilibrium. Economically, $\lambda[\phi(H,H) - \phi(H,L)] \leqslant r\phi(H,L)$ can be interpreted as "weak complementarities when high-skilled"—if one is a high-skilled person, the marginal gain from being matched to a high-skilled person relative to being matched to a low-skilled person are sufficiently low. The second condition, $\lambda[\phi(L,H)-\phi(L,L)] \leqslant r\phi(L,L)$, can be interpreted as limited gains from matching with a high-skilled person if one is low-skilled himself. Taken together, I interpret these as saying "individual skills matter more." However, it must be noted that it is straightforward to construct examples of environments where $\phi(\cdot,\cdot)$ is supermodular and satisfy the above two conditions.[8] Therefore, as in Smith (2006), supermodularity is not enough to guarantee existence of an assortative matching, let alone its uniqueness. In fact, Atakan et al. (2024) showed recently that if the match function is supermodular, then an assortative matching emerges in equilibrium in the TU environment with search frictions even with skill acquisition. However, as per their definition, even an AM equilibrium is an example of an assortative matching.

The proof of Proposition 4 uses the condition that $|\Delta|$ is sufficiently high to obtain the monotonicity of $f(\cdot)$ which, eventually, establishes the uniqueness of an asymmetric equilibrium. Without this condition, it is not difficult to numerically construct $f(\cdot)$ that has multiple interior roots when $C(\cdot)$ is quadratic. However, such constructions seem to require that $\Delta + \Delta_h \leqslant 0$, which contradictions As-

[8] For instance, $\{\phi(H,H) = 15, \phi(H,L) = 8, \phi(L,H) = 2, \phi(L,L) = 1, C(x) = x + c, c = 5, r = 1, \lambda = 1\}$ is one such environment.

sumption 1. Nevertheless, trying to obtain an example of multiple asymmetric equilibria in an environment consistent with Assumption 1 is mostly a distraction. The main purpose of much of the exercise here is to demonstrate the mechanisms that give rise to asymmetric equilibria and illustrate their robustness in some sense through microfounded utility functions. In this regard, if the cost function were affine, we obtain an even sharper and complete characterization without requiring that Δ be sufficiently high. We now turn to this next.

5.1. Affine costs

In this section, we specialize to affine costs. Here, we obtain sharper characterizations of the asymmetric equilibria without requiring that Δ be sufficiently high (in absolute value). To this end, let us specialize to affine costs: $C(x) = px + c$ for some $p > 0, c \geqslant 0$. Let us normalize $p = 1$ by scaling payoffs. Define the following two objects that play an important role Proposition 5 that characterizes equilibrium in this environment.

$$\bar{x} := \frac{\lambda}{r+\lambda}\Delta_h - c$$

$$\underline{x} := \frac{\lambda}{r+\lambda}[\Delta + \Delta_h] - c$$

$$x_{sym} := \frac{\frac{\lambda}{r+\lambda}\Delta_h - c}{1 - \frac{\lambda}{r+\lambda}\Delta}$$

Notice that $\Delta \leqslant 0 \implies \bar{x} \geqslant \underline{x}$. Moreover, $\bar{x} \geqslant x_{sym} \geqslant \underline{x}$ whenever $\underline{x} \leqslant 1$ and $\Delta \leqslant 0$.

Definition 1: *Define the following two strategy profiles.*

1. *A strategy profile is a "full investment from one side (FIOS)" strategy profile if it is given by $(1, \underline{x})$, i.e., all the men invest in skill acquisition, while women with a type $x \leqslant \underline{x}$ do so.*
2. *A strategy profile is a "no investment from one side (NIOS)" strategy profile if it is given by $(0, \bar{x})$, i.e., no man invests in skill acquisition, while women with a type $x \leqslant \bar{x}$ do so.*

As before, we do not distinguish between the permuted strategy profiles and label them as the same strategy profile.

Proposition 5: *Suppose that $C(\cdot)$ is affine and $\phi(\cdot,\cdot)$ is submodular ($\Delta \leqslant 0$). Then, the following hold.*

1. *At most one symmetric AM exists.*
2. *Except for the knife-edge cases, there is at most one (up to permutation) asymmetric AM equilibrium. In particular, either no asymmetric equilibrium exists, or a FIOS is the unique asymmetric equilibrium, or a NIOS is the unique asymmetric equilibrium.*

3. *Whenever two equilibria exist, they are Pareto incomparable.*
4. *If* $\underline{x} \leqslant 1, 1 + \frac{\lambda}{r+\lambda}\Delta \leqslant 0$, *and a symmetric equilibrium exists, then a FIOS equilibrium exists.*
5. *If a NIOS equilibrium exists and* $\frac{\lambda}{r+\lambda} \leqslant 1$, *then a symmetric equilibrium exists.*

Proposition 5 throws light on the structure of the equilibrium set. In particular, it shows that there can be at most two equilibria in any given environment. However, we are yet to understand when does an asymmetric equilibrium exists. More importantly, Proposition 5 is silent on whether we can ever have an environment featuring a unique equilibrium with that equilibrium being asymmetric. Now, we will demonstrate a constructive approach of doing so. To this end, the following numerical example would best illustrate the underlying mechanism.

Example 1: *Suppose that* $c = 2, r = \lambda = 1, \phi(L, L) = 1$. *Consider the following two environments.*

(i) $\phi(H, H) = 7, \phi(H, L) = 6, \phi(L, H) = 3.$

(ii) $\widehat{\phi}(H, H) = 11, \widehat{\phi}(H, L) = 10, \widehat{\phi}(L, H) = 6.$

■

In Example 1, it is straightforward to see that in environment (i), the unique equilibrium is symmetric and has $x_{sym} = \frac{1}{3}$. On the other hand, in environment (ii), the unique equilibrium is the FIOS one with $\underline{x} = 0.5$. In particular, the symmetric strategy profile with $x_{sym} = \frac{5}{6}$ is not an equilibrium. Notice that the difference between the two environments is in payoffs where at least one partner is highly skilled. I interpret this as an increasing skill-premium. For instance, if the market wages for a highly skilled person rise, this would bring benefits to both the members of the household: even the ones who are themselves not highly-skilled. Of course, a highly-skilled individual himself/herself benefits more than if only the spouse were highly-skilled. This example is not pathological in the sense made precise by Proposition 6. It shows that, essentially for any $\phi(L, L)$ and any environment with a low skill premium, we can construct an environment with a higher skill premium (formally defined below), wherein we move from a unique equilibrium being symmetric to it being asymmetric.

Definition 2: *We say that an economy* $\widehat{\boldsymbol{\phi}} := \langle \widehat{\phi}(H, H), \widehat{\phi}(H, L), \widehat{\phi}(L, H), \widehat{\phi}(L, L)\rangle$ *exhibits a higher skill premium relative to* $\boldsymbol{\phi} := \langle \phi(H, H), \phi(H, L), \phi(L, H), \phi(L, L)\rangle$ *if,* $\widehat{\phi}(H, H) \geqslant \phi(H, H), \widehat{\phi}(H, L) \geqslant \phi(H, L), \widehat{\phi}(L, H) \geqslant \phi(L, H)$ *and* $\widehat{\phi}(L, L) = \phi(L, L)$.

Proposition 6: *Fix* r, λ, c. *Suppose that an economy* $\boldsymbol{\phi}$ *with* $\Delta \leqslant 0$ *has* $(i) 1 - \frac{\lambda}{r+\lambda}(\phi(H, H) - \phi(L, L)) > 0$, $(ii) \frac{r}{\lambda(\phi(H,H)-\phi(L,L))} > 1$, *and* $(iii) \frac{\lambda}{r+\lambda}\Delta_h - c \in (0, 1)$.[9] *Then, the following hold:*

[9] $\frac{\lambda}{r+\lambda}\Delta_h - c \in (0, 1)$ is not essential to the proof but simplifies some of the casework making the argument more transparent.

1. *The economy $\boldsymbol{\phi}$ has a unique equilibrium with it being symmetric.*
2. *There exists an economy $\widehat{\boldsymbol{\phi}}$ which exhibits a higher skill premium relative to $\boldsymbol{\phi}$ such that $\widehat{\boldsymbol{\phi}}$ features a unique equilibrium with it being asymmetric.*

While uniqueness of an asymmetric equilibrium delivered by part 2 of Proposition 6 is reassuring—for it obviates the need to invoke an equilibrium selection argument—there is no reason to view asymmetric equilibria as pathological if multiple equilibria exist. One can construct examples where we go from a unique equilibrium being symmetric to two equilibria—one symmetric and one NIOS—as the skill premium goes up. More importantly, as the following section shows, such instances can be further microfounded by viewing the household members' problem as a non-cooperative game in the spirt of Chiappori (1988). We now turn to this next.

6. Microfoundation from household maximization in NTU

Thus far, we have established that when $\phi(\cdot, \cdot)$ is submodular, we may have asymmetric equilibria even when starting from a symmetric environment: neither does the matchig technology nor do to the payoffs depend on the gender identity in any way. However, it begs the question as to whether payoff functions that support asymmetric equilibria can arise from household optimization. To this end, I present a household optimization problem as a non-cooperative game along the lines of Chiappori (1988). It serves to demonstrate two things. First, it shows how submodular payoff functions can arise from household optimization. Second, it shows how changes in the market wages for high-skilled workers can lead society to asymmetric equilibria creating "skill inequality" between genders.

Let us work backwards to see the idea clearly. The households' problem is the following. A household consists of a man m and a woman w with skills $s(m)$ and $s(w)$. Wages in the market depend only on the skill and not on the gender.[10] The households decide how to split their two units of time—one unit for each member—between labor market work and household work. Members' allocation is a result of a non-cooperative game. Members of the household pool their total income and buy some consumption from it. There are also several household chores that, if not done, bring disutility to the household. Therefore, members need to allocate some fraction of their two units of time to the household chores. Neither member wishes to perform these chores and would rather work in the outside labor market. The goal of this section is to demonstrate how increases in wages for high-skilled labor can fundamentally alter the composition of skill acquisition of choices in equilibrium. I will illustrate this using a simple parametric example. Consider a

[10] As would be clear, a gender pay gap as it exists in reality can only make it easier to construct equilibria where women invest less in skill acquisition.

household (m, w). Let the skill of member i be s_i, The market wages for skill s is t_s. Therefore, the wage of a member i is$t_i := t_{s_i}$. If a member $i's$ allocation to the outside labor market is e_i, then the net household income is $y(s, e) := t \cdot e$, where $t := (t_m, t_w)$ and $e := (e_m, e_w)$ are the wage and effort vectors. The quality of the household work as a result of their combined effort at home is $h(e) := 2 - e_i - e_j$. The utility of a member i is given by,

$$U(s_i, s_j, e_i, e_j) = u(y(s, e), h(e)) - g(1 - e_i).$$

Essentially, the households treat both, their income and the level of household chores as public goods, while the cost, $g(\cdot)$, of their work at home is private. We assume that $u_1 > 0, u_{11} \leqslant 0, u_2 > 0, u_{22} \leqslant 0$. Moreover, the cost $g(\cdot)$ is assumed to be differentiable and convex.

Members choose their allocations, (e_i, e_j) in a non-cooperative manner taking the other person's choice as given. Let us write the agents' first order condition (assuming interior allocation choices):

$$\begin{aligned} u_1(y(s,e), h(e))t_i - u_2(y(s,e), h(e)) + g'(1 - e_i) = 0 \\ \implies u_1(y(s,e), h(e))(t_i - t_j) = g'(1 - e_j) - g'(1 - e_i) \end{aligned} \tag{1}$$

Therefore, if the players' choices are interior in equilibrium, then $t_i > t_j \implies e_i > e_j$ whenever $g(\cdot)$ is strictly convex.[11] Thus, if the household consists of two members with unequal skills—and therefore unequal wages—then we will have specialization in equilibrium: the more skilled member will work more in the professional market than the one with lower skills. This observation is at the core of the following numerical example that illustrates how increasing wages for high-skilled workers can give rise to asymmetric equilibria.

Example 2: *Consider a household consisting of (m, w). The utility of member i is $U(y(s, e), h(e)) = [\alpha \log(t \cdot e) + (1 - \alpha) \log(2 - e_m - e_w)] - \frac{1}{2}(1 - e_i)^2$, where $K = 8$ and $\alpha = 0.6$. Let $t_l = 2$. Consider the wage for high-skills going from $t_h = 3$ to $\hat{t}_h = 6.5$. The cost of acquiring skills for an agent with type x is $C(x) = c + x$, where $c = 0.25$.* ■

When two members of skill s and s' match, they choose their effort choices non-cooperatively. Using (1), we solve this numerically for each pair of skills $(h, h), (h, l), (l, h), (l, l)$ to obtain $\phi(H, H), \phi(H, L), \phi(L, H), \phi(L, L)$ corresponding to $t_h = 3$ and $\widehat{\phi}(H, H), \widehat{\phi}(H, L), \widehat{\phi}(L, H), \widehat{\phi}(L, L)$ corresponding to $t_h = 6.5$. We provide a detailed working of this in the appendix, but the parameters are as

[11] In fact, this also implies that if $g(\cdot)$ were affine, then the allocation choices cannot be interior if the wages are unequal.

follows:

$$\phi(H,H) = 5.3547, \phi(H,L) = 5.2733, \phi(L,H) = 4.7733, \phi(L,L) = 3.4085, \text{ and}$$
$$\widehat{\phi}(H,H) = 9.0660, \widehat{\phi}(H,L) = 8.9847, \widehat{\phi}(L,H) = 8.4847, \widehat{\phi}(L,L) = 3.4085.$$

Finally, in the environment with $t_h = 3$, the unique equilibrium is symmetric with $x_{sym} = 0.4157$. In contrast, when $t_h = 6.5$, $\widehat{x}_{sym} = 0.7257$ no longer constitutes an equilibrium. For this strategy profile, (AM: L-L) fails: low-skilled agents prefer to reject and wait for the high-skilled partner rather than accepting a low-skilled one. The unique equilibrium here is the FIOS one, with $\widehat{\underline{x}} = 0.0407$. Thus, in response to increasing wages, we may go from a situation where 57% of each gender were investing in acquiring high skills to one where only 4% of one gender, say men, invest in skill acquisition, while all the women invest in skill acquisition. Thus, despite no asymmetry in the payoff or technological features of the underlying environment, the equilibrium outcomes can exhibit stark asymmetries between two genders as the skill premium rises.

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A. Appendix: Proofs

Proof of Proposition 1. First, let us prove that the acceptance strategies constitute an equilibrium in the continuation game—the game after choosing the investment. In this game, we say that an agent type is (γ, s) to mean that her gender is γ and skill is s. Then, given $(\zeta^{\gamma'}, \alpha^{\gamma'})$, agent chooses $\alpha^{\gamma}(H)$ to maximize $V^{\gamma}(H, \alpha^{\gamma}; \zeta^{\gamma'}, \alpha^{\gamma'})$ and $\alpha^{\gamma}(L)$ to maximize $V^{\gamma}(L, \alpha^{\gamma}; \zeta^{\gamma'}, \alpha^{\gamma'})$. Let (α, ζ) denote $(\alpha^{\gamma}(H), \alpha^{\gamma}(L), \zeta^{\gamma})_{\gamma \in \{m,w\}}$. Consider the best-response map

$$T : [0,1]^4 \times [0,1]^2 \to [0,1]^2$$
$$(\alpha, \zeta) \mapsto \left(\underset{\alpha^{\gamma}(H) \in [0,1]}{\operatorname{argmax}} V^{\gamma}(H, \alpha^{\gamma}; \zeta^{\gamma'}, \alpha^{\gamma'}), \underset{\alpha^{\gamma}(L) \in [0,1]}{\operatorname{argmax}} V^{\gamma}(L, \alpha^{\gamma}; \zeta^{\gamma'}, \alpha^{\gamma'}) \right)_{\gamma \in \{m,w\}}.$$

Notice that, for any $(\zeta^{\gamma'}, \alpha^{\gamma'})$, $V^{\gamma}(\cdot, \alpha^{\gamma}; \zeta^{\gamma'}, \alpha^{\gamma'})$ continuous, quasi-concave in $\alpha^{\gamma'}$. Thus, $T(\alpha, \zeta)$ is compact, convex-valued for every fixed $\zeta^{\gamma'}$. By Berge's maximum theorem, $T(\alpha, \zeta)$ is upper hemicontinuous. Therefore, by Kakutani's fixed-point theorem, for every ζ, $\exists$ a fixed point, i.e., an equilibrium $(\alpha^{\gamma}_{*}(\zeta))_{\gamma \in \{m,w\}}$ in the continuation game given the decisions that induce ζ.

Now, let us turn to the investment decisions. An agent of type x and gender γ invests if $V^{\gamma}(H, \alpha^{\gamma *}; \zeta^{\gamma'}, \alpha^{\gamma'}_{*}) - C(x) \geqslant V^{\gamma}(L, \alpha^{\gamma}_{*}; \zeta^{\gamma'}, \alpha^{\gamma'}_{*})$. Since $C(\cdot)$ is increasing, there is a cutoff x^{γ}_{*} such that agents of gender γ and type above x^{γ}_{*} stay low-skilled, while the ones below acquire high skills. Therefore, in any equilibrium, the investment strategies are given by two cutoff types $\hat{x}_{\gamma}$, one for each γ. Thus, we use ζ and $\hat{X}$ interchangeably henceforth.[12] For every such strategy, we have a set of continuation equilibria. Let

$$\mathcal{A}(\hat{X}) := \{\alpha : \alpha \in T(\alpha, \hat{X})\}$$

be the set of continuation equilibria given $\hat{X}$. Since $T(\cdot, \cdot)$ is upper hemicontinuous, $\mathcal{A}(\hat{X})$ is compact and upper hemicontinuous. Define,

$$\begin{aligned} h^{\gamma}(\alpha, x^{\gamma}, \hat{X}) :=& V^{\gamma}(H, \alpha^{\gamma}; \hat{x}_{\gamma'}, \alpha^{\gamma'}) - C(x^{\gamma}) - V^{\gamma}(L, \alpha^{\gamma}; \hat{x}_{\gamma'}, \alpha^{\gamma'}) \\ W(\hat{X}) :=& \{x : h^{\gamma}(\alpha, x^{\gamma}, \hat{X}) = 0 \text{ for some } \alpha \in \mathcal{A}(\hat{X}), \text{ and } \gamma \in \{m,w\}.\} \end{aligned}$$

Let us argue that $W(\cdot)$ is compact-valued and upper hemicontinuous. First, consider any $x_n \in W(\hat{X})$ that converges to x. Then, $\exists$ $\alpha_n \to \alpha$ (by passing on to a subsequence), such that $h^{\gamma}(\alpha_n, x^{\gamma}_n, \hat{X}) = 0$ for all γ and n. By continuity of $h(\cdot)$, $h^{\gamma}(\alpha, x^{\gamma}, \hat{X}) = 0$ for all γ. Since $\mathcal{A}(\cdot)$ is upper hemicontinuous, $\alpha \in \mathcal{A}(\hat{X})$. Therefore, $x \in W(\hat{X})$. Now, let us argue that $W(\cdot)$ is upper hemicontinuous. To this end, let $\hat{X}_n \to \hat{X}$ such that $x_n \to x$ and $x_n \in W(\hat{X}_n)$ for each n. Therefore, $\exists \alpha_n$ such that $h^{\gamma}(\alpha_n, x^{\gamma}_n, \hat{X}_n) = 0$ for all γ and n. Again, $\alpha_n \to \alpha \in \mathcal{A}(\hat{X})$ due to the

[12] Vectors (ζ^m, ζ^w) and $(\hat{x}_m, \hat{x}_w)$ are denoted by ζ and $\hat{X}$ respectively.

upper hemicontinuity of $\mathcal{A}(\cdot)$. Finally, by the continuity of $h(\cdot)$, $h^\gamma(\alpha, x^\gamma, \widehat{X}) = 0$. Therefore, $x \in W(\widehat{X})$. Therefore, by Kakutani's fixed point theorem, $\exists$ a fixed point, i.e., a $\widehat{X}_*$ such that $\widehat{X}_* \in W(\widehat{X}_*)$, an equilibrium. □

A.1. Proof of Proposition 2

Proof of Proposition 2. From (PAM: Assortativity), we have that

$$\frac{\lambda \widehat{x}_{\gamma'} \phi(H,H)}{r + \lambda \widehat{x}_{\gamma'}} \geqslant \phi(H,L)$$

The LHS is increasing in $\widehat{x}_{\gamma'}$, and hence, is maximized at 1. Substituting $\widehat{x}_{\gamma'} = 1$, we get that $V^\gamma(H; \widehat{X})$ is maximized at $\frac{\lambda\phi(H,H)}{r+\lambda}$. Thus, if $\lambda[\phi(H,H) - \phi(H,L)] < r\phi(H,L)$, then (PAM: Assortativity) cannot be satisfied for any $\widehat{X}$ establishing the first part of the Proposition.

For the the symmetry of a PAM whenever it exists, notice that from (PAM: Invest), we have,

$$\begin{aligned}
V_P^m(H;\widehat{X}) - C(\widehat{x}_m) &= V_P^m(L;\widehat{X}) \quad \text{and} \\
V_P^w(H;\widehat{X}) - C(\widehat{x}_w) &= V_P^w(L;\widehat{X}). \\
\implies [V_P^m(H;\widehat{X}) - V_P^m(L;\widehat{X})] - [V_P^w(H;\widehat{X}) - V_P^w(L;\widehat{X})] &= C(\widehat{x}_m) - C(\widehat{x}_w)
\end{aligned}$$

Suppose that $\widehat{x}_m \neq \widehat{x}_w$. Say, wlog, $\widehat{x}_m > \widehat{x}_w$. Notice that, from (PAM: Values)

$$\begin{aligned}
&\left[V_P^m(H;\widehat{X}) - V_P^m(L;\widehat{X})\right] - \left[V_P^w(H;\widehat{X}) - V_P^w(L;\widehat{X})\right] \\
&= \left[\frac{\lambda \widehat{x}_w \phi(H,H)}{r + \lambda \widehat{x}_w} - \frac{\lambda(1-\widehat{x}_w)\phi(L,L)}{r + \lambda(1-\widehat{X}^w)}\right] \\
&- \left[\frac{\lambda \widehat{x}_m \phi(H,H)}{r + \lambda \widehat{x}_m} - \frac{\lambda(1-\widehat{x}_m)\phi(L,L)}{r + \lambda(1-\widehat{X}^m)}\right]
\end{aligned}$$

Let $h(x) := \left[\frac{\lambda x \phi(H,H)}{r+\lambda x} - \frac{\lambda(1-x)\phi(L,L)}{r+\lambda(1-x)}\right]$. Notice that $h(\cdot)$ is increasing. Therefore,

$$\left[V_P^m(H;\widehat{X}) - V_P^m(L;\widehat{X})\right] - \left[V_P^w(H;\widehat{X}) - V_P^w(L;\widehat{X})\right] = h(\widehat{x}_w) - h(\widehat{x}_m) < 0 \quad \text{if } \widehat{x}_m > \widehat{x}_w.$$

At the same time, $C(\widehat{x}_m) - C(\widehat{x}_w) > 0$ whenever $\widehat{x}_m > \widehat{x}_w$. Therefore,

$$0 < C(\widehat{x}_m) - C(\widehat{x}_w) = [V_P^m(H;\widehat{X}) - V_P^m(L;\widehat{X})] - [V_P^w(H;\widehat{X}) - V_P^w(L;\widehat{X})] < 0,$$

a contradiction. □

A.2. Proof of Proposition 3

Proof of Proposition 3. Let us split the analysis into three cases:

Case 1: $\widehat{X} \in (0,1)^2$. Suppose that $\widehat{x}_m \neq \widehat{x}_w$. From (AM: Invest), we have,

$$\begin{aligned} V_A^{\gamma}(H;\widehat{X}) - V_A^{\gamma}(L;\widehat{X}) &= \frac{\lambda}{r+\lambda}\Big[\widehat{x}_{\gamma'}(\underbrace{\phi(H,H)+\phi(L,L)-\phi(H,L)-\phi(L,H)}_{\geqslant 0}) \\ &\quad + \underbrace{\phi(H,L)-\phi(L,L)}_{\geqslant 0}\Big] \\ &= C(\widehat{x}_{\gamma}). \end{aligned}$$

Therefore,

$$\begin{aligned} &\big[V_A^m(H;\widehat{X}) - V_A^m(L;\widehat{X})\big] - \big[V_A^w(H;\widehat{X}) - V_A^w(L;\widehat{X})\big] \\ &= \frac{\lambda}{r+\lambda}(\widehat{x}_w - \widehat{x}_m)\Delta \\ &= C(\widehat{x}_m) - C(\widehat{x}_w) \end{aligned}$$

Therefore, if $\widehat{x}_m > \widehat{x}_w$, then $LHS < 0 < RHS$, a contradiction.

Case 2: $\widehat{x}_m \in [0,1)$ and $\widehat{x}_w = 1$. Then,

$$\begin{aligned} V_A^m(H;\widehat{X}) - V_A^m(L;\widehat{X}) &= \frac{\lambda}{r+\lambda}[\widehat{x}_w\Delta + \Delta_h] \leqslant C(\widehat{x}_m) \\ V_A^w(H;\widehat{X}) - V_A^w(L;\widehat{X}) &= \frac{\lambda}{r+\lambda}[\widehat{x}_m\Delta + \Delta_h] \geqslant C(\widehat{x}_w) \end{aligned}$$

Therefore,

$$\begin{aligned} &[V_A^m(H;\widehat{X}) - V_A^m(L;\widehat{X})] - [V_A^w(H;\widehat{X}) - V_A^w(L;\widehat{X})] \\ &= \frac{\lambda}{r+\lambda}(\widehat{x}_w - \widehat{x}_m)\Delta = \frac{\lambda}{r+\lambda}(1-\widehat{x}_m)\Delta \leqslant C(\widehat{x}_m) - C(1) \end{aligned}$$

However, $LHS \geqslant 0 > RHS$, a contradiction.

Case 3: $\widehat{x}_m \in (0,1]$ and $\widehat{x}_w = 0$. Then,

$$\begin{aligned} V_A^m(H;\widehat{X}) - V_A^m(L;\widehat{X}) &= \frac{\lambda}{r+\lambda}[\widehat{x}_w\Delta + \Delta_h] \geqslant C(\widehat{x}_m) \\ V_A^w(H;\widehat{X}) - V_A^w(L;\widehat{X}) &= \frac{\lambda}{r+\lambda}[\widehat{x}_m\Delta + \Delta_h] \leqslant C(\widehat{x}_w) \end{aligned}$$

Therefore,

$$\begin{aligned}&[V_A^m(H;\widehat{X}) - V_A^m(L;\widehat{X})] - [V_A^w(H;\widehat{X}) - V_A^w(L;\widehat{X})]\\&= \frac{\lambda}{r+\lambda}(\widehat{x}_w - \widehat{x}_m)\Delta = -\frac{\lambda}{r+\lambda}\widehat{x}_m\Delta \geqslant C(\widehat{x}_m) - C(0)\end{aligned}$$

However, $LHS \leqslant 0 < RHS$, a contradiction. □

A.3. Proof of Proposition 4

Proof of Proposition 4. Let us start with symmetric equilibria and argue that it is unique whenever one exists. Suppose that $\widehat{X}$ is a symmetric equilibrium. We have the following necessary conditions for an equilibrium:

$$V_A^\gamma(H;\widehat{X}) - V_A^\gamma(L;\widehat{X}) = \frac{\lambda}{r+\lambda}[\Delta\widehat{x} + \Delta_h]\begin{cases}\geqslant C(1) & \text{if } \widehat{x} = 1\\ = C(\widehat{x}) & \text{if } \widehat{x} \in (0,1)\\ \leqslant C(0) & \text{if } \widehat{x} = 0.\end{cases}$$

Notice that $\frac{\lambda}{r+\lambda}[\Delta\widehat{x} + \Delta_h] - C(\widehat{x})$ is decreasing in $\widehat{x}$. Therefore, exactly one of the above three conditions will hold, i.e., if a symmetric equilibrium exists it is unique. As for its existence, let us revisit (AM: H-L) and (AM: L-L).

$$\begin{aligned}V_A^\gamma(H;\widehat{X}) &= \frac{\lambda}{r+\lambda}[\widehat{x}\phi(H,H) + (1-\widehat{x})\phi(H,L)] \leqslant \phi(H,L)\\ V_A^\gamma(L;\widehat{X}) &= \frac{\lambda}{r+\lambda}[\widehat{x}\phi(L,H) + (1-\widehat{x})\phi(L,L)] \leqslant \phi(L,L)\end{aligned}$$

Notice that the LHS of both the above are increasing in $\widehat{x}$. Therefore, if $\frac{\lambda}{r+\lambda}\phi(H,H) \leqslant \phi(H,L)$ and $\frac{\lambda}{r+\lambda}\phi(L,H) \leqslant \phi(L,L)$, then (AM: H-L) and (AM: L-L) are satisfied for any $\widehat{x}$. Thus, we have a unique symmetric equilibrium in this case.

Let us now turn to asymmetric equilibria. Suppose an interior asymmetric equilibrium exists. Let $\widehat{X} = (\widehat{x}_m, \widehat{x}_w) \in (0,1)^2$ be an asymmetric equilibrium. Then, (AM: Invest) yields,

$$\begin{aligned}\frac{\lambda}{r+\lambda}[\widehat{x}_w\Delta + \Delta_h] &= C(\widehat{x}_m),\\ \frac{\lambda}{r+\lambda}[\widehat{x}_m\Delta + \Delta_h] &= C(\widehat{x}_w).\end{aligned} \tag{2}$$

Let $a := \frac{\lambda}{r+\lambda}\Delta$ and $b := \frac{\lambda}{r+\lambda}\Delta_h$. Then, $\widehat{x}_m$ solves

$$f(x) := aC^{-1}(ax+b) + b - C(x) = 0$$

where, $C^{-1}(y) = 0$ if $y < C(0)$. Notice that if $f(x) > (<)\ 0$, then an agent of type x would strictly prefer acquiring (not acquiring) high skills. If x and $C^{-1}(ax+b)$

are both interior, then $f(x) = 0$ is a candidate equilibrium. Differentiating $f(\cdot)$,

$$f'(x) = \frac{a^2}{C'(C^{-1}(ax + b))} - C'(x)$$

Notice that $C'(C^{-1}(ax + b))$ is decreasing. Therefore, if $a^2 \geqslant C'(1)C'(C^{-1}(b))$, then $f'(\cdot) > 0$ over $[0, x^*]$ where x^* is given by $ax^* + b = C(0)$. Therefore, there exists at most one $\widehat{x}_m < x^*$ such that $f(\widehat{x}_m) = 0$. The corresponding $\widehat{x}_w$ is given by $C^{-1}(a\widehat{x}_m + b)$. By symmetry, if $(\widehat{x}_m, \widehat{x}_w)$ solves (2), then so does $(\widehat{x}_w, \widehat{x}_m)$. This contradicts the uniqueness of $\widehat{x}_m$ if $\widehat{x}_m \neq \widehat{x}_w$.

Now, let us turn to the possibility of asymmetric equilibria with one of $\widehat{x}_m$ or $\widehat{x}_w \in \{0, 1\}$. Let us first start with candidate equilibria of the form $(\widehat{x}_m, 1)$ where $\widehat{x}_m \in (0, 1)$. Then, we have,

$$\begin{aligned} C(\widehat{x}_m) &= a + b \\ C(1) &\leqslant a\widehat{x}_m + b. \end{aligned}$$

However, if $|a|$ is sufficiently large, then $a + b \leqslant 0 \implies \widehat{x}_m = 0$, a contradiction. Therefore, no such equilibria can exist for a sufficiently large $|\Delta|$. The other candidate asymmetric equilibria are of the form $(\widehat{x}_m, 0)$ (and of course its mirror image) with $\widehat{x}_m \in (0, 1)$. For this to be an equilibrium, we need,

$$\begin{aligned} C(\widehat{x}_m) &= b \\ C(0) &\geqslant a\widehat{x}_m + b. \end{aligned}$$

Notice that whenever the above is satisfied, the only relevant incentive constraint for this to be an equilibrium is (AM: L-L). That is, say $\widehat{x}_m \in (0, 1)$ and $\widehat{x}_w = 0$. Then, a high-skilled man can only meet a low-skilled woman on path. Therefore, we only need to check that a low-skilled woman does not want to reject a low-skilled man and wait to be matched with a high-skilled man. Whenever this constraint is satisfied, two asymmetric equilibria—$(\widehat{x}_m, 0)$ and $(0, \widehat{x}_m)$—exist. □

A.4. Proof of Proposition 5

Proof of Proposition 5. **Proof of 1:** Recall that the interior equilibria are characterized by (2):

$$\begin{aligned} \frac{\lambda}{r + \lambda}[\widehat{x}_w \Delta + \Delta_h] &= C(\widehat{x}_m) = \widehat{x}_m + c, \\ \frac{\lambda}{r + \lambda}[\widehat{x}_m \Delta + \Delta_h] &= C(\widehat{x}_w) = \widehat{x}_w + c. \end{aligned}$$

Therefore, a candidate symmetric equilibrium has $\widehat{x}_m = \widehat{x}_w = \frac{\frac{\lambda}{r+\lambda}\Delta_h - c}{1 - \frac{\lambda}{r+\lambda}\Delta} =: \widehat{x}_{sym}$. Since these are two simultaneous linear equations, the above solution is unique except in the knife-edge cases. If $\widehat{x}_{sym} \in (0,1)$, then it is the unique symmetric equilibrium if it satisfies (AM: H-L) and (AM: L-L).

Proof of 2: If an asymmetric equilibrium exists, at least one of $\widehat{x}_m, \widehat{x}_w \in \{0,1\}$. Suppose that an asymmetric equilibrium exists with $\widehat{x}_m = 1$. Then, a threshold type of woman who invests given that $\widehat{x}_m = x$ satisfies the following:

$$\widehat{x}_w(x) = \begin{cases} 0 & \text{if } \frac{\lambda}{r+\lambda}[\Delta x + \Delta_h] - c \leqslant 0 \\ 1 & \text{if } \frac{\lambda}{r+\lambda}[\Delta x + \Delta_h] - (1+c) \geqslant 0 \\ \frac{\lambda}{r+\lambda}[\Delta x + \Delta_h] - c & \text{otherwise.} \end{cases}$$

Simply, this type is interior if $\frac{\lambda}{r+\lambda}[\Delta x + \Delta_h] - (y+c)$ (as a function of y) has a root in $(0,1)$. Otherwise, it is $\widehat{x}_w = 1$ if this function is nonnegative on $[0,1]$ and 0 if it is nonpositive on $[0,1]$. Since $\frac{\lambda}{r+\lambda}[\Delta x + \Delta_h] - (y+c)$ is strictly decreasing in y, $\widehat{x}_w(x)$ is unique for any given x. Therefore, we have only two possible asymmetric equilibria $(0, x_1)$ and $(1, x_2)$ (again, up to permutation). By the monotonicity (in y) of $\frac{\lambda}{r+\lambda}[\Delta x + \Delta_h] - (y+c)$, $1 > x_1 > x_2 > 0$.[13] Suppose that $(1, x_2)$ is an equilibrium. Then,

$$\frac{\lambda}{r+\lambda}[\Delta + \Delta_h] - c \geqslant 0 \implies \frac{\lambda}{r+\lambda}[\Delta x_1 + \Delta_h] - c > 0.$$

Therefore, $(0, x_1)$ cannot be a best response. Similarly, if $(0, x_1)$ is an equilibrium, then,

$$\frac{\lambda}{r+\lambda}[\Delta x_1 + \Delta_h] - c < 0 \implies \frac{\lambda}{r+\lambda}[\Delta + \Delta_h] - (x_2 + c) < 0$$

for any $x_2 \in [0,1]$. Therefore, $(1, x_2)$ cannot be an equilibrium.

Proof of 3: The Pareto ranking of the equilibria is straightforward. Recall that $\bar{x} \geqslant \underline{x}$. Let us compare (x_{sym}, x_{sym}) with $(1, \underline{x})$ first. Given that $(1, \underline{x})$ is an equilibrium, $\frac{\lambda}{r+\lambda}[\Delta + \Delta_h] - c \leqslant 1$. It is easy to check that this implies that $x_{sym} \geqslant \underline{x}$. Therefore, men with $x_m \leqslant x_{sym}$ prefer the symmetric equilibrium to the FIOS one, while all the women prefer the FIOS one (with all men investing). Similarly, between the symmetric and NIOS equilibrium $(0, \bar{x})$, all the men prefer the NIOS equilibrium (since $\bar{x} \geqslant x_{sym}$) while women with type $x_w \leqslant \bar{x}$ strictly prefer the symmetric one. Since the FIOS and NIOS equilibria cannot co-exist, a Pareto comparison between them is unnecessary.

Proof of 4: Suppose that $\underline{x} \leqslant 1$ and a symmetric equilibrium exists. Since $\Delta \leqslant 0$, this implies that $\underline{x} \geqslant 0$. Therefore, we need to check for (AM: H-L) and

[13] Since we have assumed that the two equilibria $(0, x_1)$ and $(1, x_2)$ is different beyond permutation, we must have $x_1 \neq 1$ and $x_2 \neq 0$.

(AM: Invest) for the FIOS equilibrium. (AM: H-L) requires,

$$\frac{\lambda}{r+\lambda}[\underline{x}\phi(H,H) + (1-\underline{x})\phi(H,L)] - \phi(H,L) \leqslant 0$$

Since $\underline{x} \leqslant x_{sym}$ and since (AM: H-L) holds for the symmetric equilibrium, we have,

$$\frac{\lambda}{r+\lambda}[x_{sym}\phi(H,H) + (1-x_{sym})\phi(H,L)] - \phi(H,L) \leqslant 0$$

Finally,

$$\begin{aligned} &\frac{\lambda}{r+\lambda}\big[\underline{x}\phi(H,H) + (1-\underline{x})\phi(H,L)\big] - \phi(H,L) \\ \leqslant\ &\frac{\lambda}{r+\lambda}\big[x_{sym}\phi(H,H) + (1-x_{sym})\phi(H,L)\big] - \phi(H,L) \leqslant 0. \end{aligned}$$

Now, let us turn to (AM: Invest). It says,

$$\begin{aligned} &\frac{\lambda}{r+\lambda}(\underline{x}\Delta + \Delta_h) \geqslant 1 + c, \\ \iff &\frac{\lambda}{r+\lambda}\Delta_h - c + \frac{\lambda}{r+\lambda}\Delta\left(\frac{\lambda}{r+\lambda}(\Delta+\Delta_h) - c\right) \geqslant 1, \\ \iff &\left(\frac{\lambda}{r+\lambda}\Delta_h - c\right)\left(1 + \frac{\lambda}{r+\lambda}\Delta\right) \geqslant \left(1 - \frac{\lambda}{r+\lambda}\Delta\right)\left(1 + \frac{\lambda}{r+\lambda}\Delta\right), \\ \iff &\left(\frac{\lambda}{r+\lambda}\Delta_h - c\right) \leqslant \left(1 - \frac{\lambda}{r+\lambda}\Delta\right), \quad \text{since } 1 + \frac{\lambda}{r+\lambda}\Delta \leqslant 0, \\ \iff &\frac{\lambda}{r+\lambda}(\Delta+\Delta_h) - c = \underline{x} \leqslant 1. \end{aligned}$$

Therefore, FIOS is an equilibrium.

Proof of 5: Suppose that a NIOS equilibrium exists. Recall that $\bar{x} \geqslant x_{sym}$. Therefore,

$$\begin{aligned} &\frac{\lambda}{r+\lambda}[x_{sym}\phi(L,H) + (1-x_{sym})\phi(L,L)] - \phi(L,L) \\ \leqslant &\frac{\lambda}{r+\lambda}[\bar{x}\phi(L,H) + (1-\bar{x})\phi(L,L)] - \phi(L,L) \leqslant 0 \end{aligned}$$

Therefore, (AM: L-L) is satisfied for the symmetric equilibrium.

Also, notice that

$$\begin{aligned}
&\frac{\lambda}{r+\lambda}[x_{sym}\phi(H,H)+(1-x_{sym})\phi(H,L)]-\phi(H,L)\\
\leqslant&\frac{\lambda}{r+\lambda}[x_{sym}\phi(L,H)+(1-x_{sym})\phi(L,L)]-\phi(L,L)\\
\Longleftrightarrow&\Delta_h\left(1-\frac{\lambda}{r+\lambda}\right)\geqslant\frac{\lambda}{r+\lambda}x_{sym}\Delta
\end{aligned}$$

The RHS is nonpositive, while the LHS is nonnegative if $\frac{\lambda}{r+\lambda}\leqslant 1$. Finally, since $\frac{\lambda}{r+\lambda}[x_{sym}\phi(L,H)+(1-x_{sym})\phi(L,L)]-\phi(L,L)\leqslant 0$ due to (AM: L-L), we have that (AM: H-L) under (x_{sym},x_{sym}), thereby completing the proof of the Proposition. □

A.5. Proof of Proposition 6

Proof of Proposition 6. Let us first prove that $\boldsymbol{\phi}$ has a unique equilibrium and that is symmetric. To this end, let us rule out a NIOS equilibrium first. (AM: Invest) says,

$$\frac{\lambda}{r+\lambda}[\Delta\bar{x}+\Delta_h]-c\leqslant\ 0$$

This ensures the optimality of no investment from one side of the market. This equation rearranges to yield,

$$\left(1+\frac{\lambda}{r+\lambda}\Delta\right)\left(\frac{\lambda}{r+\lambda}\Delta_h-c\right)\leqslant 0$$

Moreover, we obviously need that $\bar{x}=\frac{\lambda}{r+\lambda}\Delta_h-c\in[0,1]$, and therefore, one necessary condition is $1+\frac{\lambda}{r+\lambda}\Delta\leqslant 0$. Notice that $|\Delta|\leqslant\phi(H,H)-\phi(L,L)$. Therefore,

$$\Delta\geqslant-(\phi(H,H)-\phi(L,L))\implies 1+\frac{\lambda}{r+\lambda}\Delta\geqslant 1-\frac{\lambda}{r+\lambda}(\phi(H,H)-\phi(L,L))\geqslant 0.$$

Therefore, (AM: Invest) cannot be satisfied for the NIOS equilibrium.

Now, let us turn to a FIOS equilibrium. (AM: Invest) for this equilibrium is,

$$\frac{\lambda}{r+\lambda}[\Delta\underline{x}+\Delta_h]\geqslant 1+c$$

Recall that $\underline{x} = \frac{\lambda}{r+\lambda}(\Delta + \Delta_h) - c$. Substituting the above, we get,

$$\begin{aligned}
&\frac{\lambda}{r+\lambda}\Delta_h - c + \frac{\lambda}{r+\lambda}\Delta\left(\frac{\lambda}{r+\lambda}(\Delta+\Delta_h) - c\right) \geqslant 1, \\
\iff &\left(\frac{\lambda}{r+\lambda}\Delta_h - c\right)\left(1+\frac{\lambda}{r+\lambda}\Delta\right) \geqslant \left(1-\frac{\lambda}{r+\lambda}\Delta\right)\left(1+\frac{\lambda}{r+\lambda}\Delta\right), \\
\iff &\left(\frac{\lambda}{r+\lambda}\Delta_h - c\right) \geqslant \left(1-\frac{\lambda}{r+\lambda}\Delta\right), \quad \text{since } 1+\frac{\lambda}{r+\lambda}\Delta > 0, \\
\iff &\frac{\lambda}{r+\lambda}(\Delta+\Delta_h) - c = \underline{x} \geqslant 1.
\end{aligned}$$

However, since $\frac{\lambda}{r+\lambda}(\Delta + \Delta_h) = \frac{\lambda}{r+\lambda}(\phi(H,H) - \phi(L,H)) < 1$, the above cannot be satisfied. Therefore, FIOS is not an equilibrium.

Finally, let us turn to proving that a symmetric equilibrium exists. First, by assumption $x_{sym} \in (0,1)$. Therefore, we need to verify (AM: H-L) and (AM: L-L) for this strategy profile. Those can be written as,

$$\begin{aligned}
\lambda x_{sym}(\phi(H,H) - \phi(H,L)) &\leqslant r\phi(H,L) \\
\lambda x_{sym}(\phi(L,H) - \phi(L,L)) &\leqslant r\phi(L,L)
\end{aligned}$$

However, $\max\{\phi(H,H)-\phi(H,L), \phi(L,H)-\phi(L,L)\} \leqslant \phi(H,H)-\phi(L,L)$. Therefore,

$$\begin{aligned}
\lambda x_{sym}(\phi(H,H) - \phi(H,L)) \leqslant \lambda x_{sym}(\phi(H,H) - \phi(L,L)) < \frac{\lambda x_{sym} r}{\lambda} \leqslant r\phi(H,L) \\
\lambda x_{sym}(\phi(L,H) - \phi(L,L)) \leqslant \lambda x_{sym}(\phi(H,H) - \phi(L,L)) < \frac{\lambda x_{sym} r}{\lambda} \leqslant r\phi(L,L).
\end{aligned}$$

where the last inequality for the above uses the fact that $\phi(H,L) \geqslant \phi(L,L) = 1$. Therefore, (AM: H-L) and (AM: L-L) are satisfied by (x_{sym}, x_{sym}), i.e., there exists a unique equilibium and is symmetric. This completes the proof of the first part of the proposition.

Now, let us turn to constructing $\widehat{\boldsymbol{\phi}}$ that exhibits a higher skill premium relative to $\boldsymbol{\phi}$ and admits a unique equilibrium, with it being asymmetric. In particular, we will choose $\widehat{\phi}(H,H), \widehat{\phi}(H,L), \widehat{\phi}(L,H)$ larger than $\phi(H,H), \phi(H,L), \phi(L,H)$ that admit a FIOS equilibrium, and it would be a unique equilibrium. Let us rewrite the constraints of a FIOS equilibrium.

$$\begin{aligned}
\frac{\lambda}{r+\lambda}(\Delta + \Delta_h) - c &\in (0,1). \\
\frac{\lambda}{r+\lambda}(\underline{x}\Delta + \Delta_h) &\geqslant 1 + c \\
\frac{\lambda}{r+\lambda}(\underline{x}\widehat{\phi}(H,H) + (1-\underline{x})\widehat{\phi}(H,L)) &\leqslant \widehat{\phi}(H,L)
\end{aligned}$$

Of the three constraints above, our construction would involve $\widehat{\phi}(H,H) \approx \widehat{\phi}(H,L)$. Therefore, the third constraint would be easily satisfied and hence, it can be ignored.

Finally, for uniqueness, we also need to ensure that (x_{sym}, x_{sym}) is not an equilibrium. Out of the two IC constraints, the particular one we would constructively violate is (AM: L-L):

$$\frac{\lambda}{r+\lambda}(x_{sym}\widehat{\phi}(L,H) + (1-x_{sym})\widehat{\phi}(L,L)) \leqslant \widehat{\phi}(L,L)$$

The above constraint would be violated if $\widehat{\phi}(L,H)$ is sufficiently large.

To this end, we will choose $\widehat{\phi}(L,H) = a$ sufficiently larger than $\widehat{\phi}(L,L) = \phi(L,L) = 1$ and $\widehat{\phi}(H,H) \approx \widehat{\phi}(H,L) = b > a$. Then, $\Delta \approx -(a-1)$, $\Delta_h \approx b-1$. First, as before, (AM: Invest) rearranges to,

$$\iff \left(\frac{\lambda}{r+\lambda}\Delta_h - c\right)\left(1 + \frac{\lambda}{r+\lambda}\Delta\right) \geqslant \left(1 - \frac{\lambda}{r+\lambda}\Delta\right)\left(1 + \frac{\lambda}{r+\lambda}\Delta\right) \tag{3}$$

Therefore, it is necessary that $1 + \frac{\lambda}{r+\lambda}\Delta < 0$. As $a \uparrow$, $|\Delta| \uparrow$, and therefore, $1 + \frac{\lambda}{r+\lambda}\Delta \downarrow$. In particular, if we choose $a = \phi(H,H)$, then $1 + \frac{\lambda}{r+\lambda}a > 0$ by assumption. Thus, $\widehat{\phi}(L,H) = a > \phi(H,H)$. We also need to ensure that $\underline{x} = k(\Delta + \Delta_h) - c = k(b-a) - c$ to be larger than 1. Given any a, this puts an upper bound on b. So long as this upper bound is satisfied, we have also ensured that that (3) is satisfied since all it requires, subject to having $1 + \frac{\lambda}{r+\lambda}\Delta < 0$ is that $\underline{x} < 1$ which we have ensured regardless by choosing a sufficiently high value of a and assuming that $\widehat{\phi}(H,H) \approx \widehat{\phi}(H,L)$. Choosing a b high enough so that $\underline{x} \in (0,1)$ would therefore guarantee that a FIOS equilibrium exists. However, for its uniqueness, we also need to choose (a,b) to violate (AM: L-L). That is, we need to choose (a,b) so that,

$$\frac{\frac{\lambda}{r+\lambda}\Delta_h - c}{1 - \frac{\lambda}{r+\lambda}\Delta}(\widehat{\phi}(L,H) - \widehat{\phi}(L,L)) > \frac{r\widehat{\phi}(L,L)}{\lambda}$$
$$\iff \frac{\frac{\lambda}{r+\lambda}(b-1) - c}{1 + \frac{\lambda}{r+\lambda}(a-1)}(a-1) > \frac{r}{\lambda}$$

Since LHS is increasing in a, $\exists$ a sufficiently large so that the above inequality is satisfied. Therefore, (x_{sym}, x_{sym}) is not an equilibrium for such an a. Subsequently, by choosing b so that $\underline{x} \in (0,1)$ we complete the construction. □